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ORIGINAL PAPER

A family of eight-step methods with vanished phase-lag and its derivatives for the numerical integration of the Schrödinger equation

Ibraheem Alolyan · T. E. Simos

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Abstract A family of tenth algebraic order eight-step methods is constructed in this paper. For this family of methods, we require the phase-lag and its first, second and third derivatives to be vanished. Three alternative methods are proposed which satisfy the above requirements. An error analysis and a stability analysis is also investigated in this paper and a comparison with other methods is also studied. The new proposed methods are applied for the numerical solution of the one dimensional Schrödinger equation. The efficiency of the new methodology is proved via theoretical analysis and numerical applications.

Keywords Numerical solution · Schrödinger equation · Multistep methods · Hybrid methods · Interval of periodicity · P-stability · Phase-lag · Phase-fitted

1 Introduction

The one-dimensional Schrödinger equation can be written as:

$$y''(x) = [l(l+1)/x^2 + V(x) - k^2]y(x). \tag{1}$$

I. Alolyan · T. E. Simos
Department of Mathematics, College of Sciences, King Saud University,
P. O. Box 2455, Riyadh 11451, Saudi Arabia

T. E. Simos
Laboratory of Computational Sciences, Department of Computer Science and Technology,
Faculty of Sciences and Technology, University of Peloponnese,
221 00, Tripolis, Greece
e-mail: tsimos.conf@gmail.com

T. E. Simos (✉)
10 Konitsis Street, Amfithea, Paleon Faliron, 175 64, Athens, Greece
e-mail: tsimos@mail.ariadne-t.gr, tsimos.conf@gmail.com

It is known from the literature that many mathematical models in theoretical physics and chemistry, material sciences, quantum mechanics and quantum chemistry, electronics etc. can be written via the above boundary value problem (see for example [1–4]).

For the above equation (1) we have the following definitions:

- The function $W(x) = l(l+1)/x^2 + V(x)$ is called *the effective potential*. This satisfies $W(x) \to 0$ as $x \to \infty$.
- The quantity k^2 is a real number denoting *the energy.*
- The quantity l is a given integer representing the *angular momentum.*
- V is a given function which denotes the *potential*.

The boundary conditions are:

$$y(0) = 0 \tag{2}$$

and a second boundary condition, for large values of x, determined by physical considerations.

In the last decades, a large research on the development of numerical methods for the approximate solution of the Schrödinger equation has been done. The construction of fast and reliable numerical methods for the efficient solution of the Schrödinger equation and related problems is the main aim and scope of this research (see for example [5–49]).

More specifically the last years:

- Phase-fitted methods and numerical methods with minimal phase-lag of Runge-Kutta and Runge-Kutta Nyström type have been developed in [10]–[35].
- In [40]–[45] exponentially and trigonometrically fitted Runge-Kutta and Runge-Kutta Nyström methods are obtained.
- Multistep phase-fitted methods and multistep methods with minimal phase-lag are developed in [50–101].
- Symplectic integrators are studied in [102–121].
- Exponentially and trigonometrically multistep methods have been developed in [122–140].
- Nonlinear methods have been studied in [141] and [142]
- Review papers have been written in [143–145]

The numerical methods for the approximate solution of the Schrödinger equation belong into two main categories:

1. Methods with constant coefficients
2. Methods with coefficients depending on the frequency of the problem.[1]

The purpose of this paper is to develop a family of new tenth algebraic order eight-step methods which have the phase-lag and its first, second and third derivatives equal to zero. We study the local truncation error and the stability of the new proposed

[1] When using a functional fitting algorithm for the solution of the radial Schrödinger equation, the fitted frequency is equal to: $\sqrt{|l(l+1)/x^2 + V(x) - k^2|}$

methods and we give also a comparative error analysis and a comparative stability analysis with other well know methods. We will apply the new proposed methods to the numerical solution of the resonance problem of the radial Schrödinger equation. The theoretical analysis and numerical applications proved the efficiency of the new methodology. We note here that some important remarks on the development of multistep methods for the approximate solution of the radial Schrödinger equation are also mentioned in this paper.

More precisely, in this paper we will study a family of implicit symmetric eight-step methods of tenth algebraic order. The logic for the development of the new family is based on the requirement of vanishing the phase-lag and its first, second and third derivatives. Based on the above logic, three methods of the above family will be developed. The difference between these methods is the selection of free parameters of the family of methods. So, in one of the them we select as free parameters the coefficients of the right hand side of the family of methods. In the other two, we select as free parameters the coefficients of the left hand side of the family of methods. For all these methods we will present a stability an error analysis. Finally, we will apply the new proposed methods to the resonance problem. This is one of the most difficult problems arising from the radial Schrödinger equation.

The paper is organized as follows:

- In Sect. 2, we present the theory of the new methodology.
- In Sect. 3, we present the development of the new family of methods.
- A comparative error analysis and its conclusions are presented in Sect. 4.
- In Sect. 5, we will investigate the stability properties of the new developed methods. In the same section a comparative analysis of the main properties of some well known methods is also presented.
- In Sect. 6, numerical results are presented.
- Remarks and conclusions are discussed in Sect. 7.
- Finally in the Appendices we present the coefficients of all the methods obtained in this paper and also the analytic expansions for the errors for several methods.

2 Phase-lag analysis of symmetric multistep methods

For the numerical solution of the initial value problem

$$q'' = f(x, q) \tag{3}$$

consider a multistep method with m steps which can be used over the equally spaced intervals $\{x_i\}_{i=0}^{m} \in [a, b]$ and $h = |x_{i+1} - x_i|$, $i = 0(1)m - 1$.

If the method is symmetric then $a_i = a_{m-i}$ and $b_i = b_{m-i}$, $i = 0(1)\lfloor \frac{m}{2} \rfloor$.

When a symmetric $2k$-step method, that is for $i = -k(1)k$, is applied to the scalar test equation

$$q'' = -\omega^2 q \tag{4}$$

a difference equation of the form

$$A_k(\mathrm{v})\, q_{n+k} + \cdots + A_1(\mathrm{v})\, q_{n+1} + A_0(\mathrm{v})\, q_n + A_1(\mathrm{v})\, q_{n-1} + \cdots + A_k(\mathrm{v})\, q_{n-k} = 0 \tag{5}$$

is obtained, where $\mathrm{v} = \omega h$, h is the step length and $A_0(\mathrm{v}), A_1(\mathrm{v}), \ldots, A_{\mathrm{k}}(v)$ are polynomials of v.

The characteristic equation associated with (5) is given by:

$$A_k(\mathrm{v})\,\lambda^k + \cdots + A_1(\mathrm{v})\,\lambda + A_0(\mathrm{v}) + A_1(\mathrm{v})\,\lambda^{-1} + \cdots + A_k(\mathrm{v})\,\lambda^{-k} = 0 \tag{6}$$

Theorem 1 *Williams and Simos* [151] *The symmetric* 2*k-step method with characteristic equation given by (6) has phase-lag order r and phase-lag constant c given by*

$$-c\,\mathrm{v}^{r+2} + O(\mathrm{v}^{r+4}) = \frac{2\,A_k(\mathrm{v})\,\cos(k\,\mathrm{v}) + \cdots + 2\,A_j(\mathrm{v})\,\cos(j\,\mathrm{v}) + \cdots + A_0(\mathrm{v})}{2\,k^2\,A_k(\mathrm{v}) + \cdots + 2\,j^2\,A_j(\mathrm{v}) + \cdots + 2\,A_1(\mathrm{v})} \tag{7}$$

The formula proposed from the above theorem gives us a direct method to calculate the phase-lag of any symmetric $2\,k$- step method.

Remark 1 The First and Second Derivatives of the phase-lag for the multistep methods are computed based on the above direct formula (7).

3 The new family of eight-step tenth algebraic order methods

3.1 First method of the family with vanished phase-lag and its first, second and third derivatives

Consider the following family of methods to integrate $q'' = f(x, q)$:

$$\sum_{i=1}^{4} a_i\,\left(q_{n+i} + q_{n-i}\right) + a_0\,q_n = h^2\left[\sum_{i=1}^{4} b_i\,\left(q''_{n+i} + q''_{n-i}\right) + b_0 q''_n\right] \tag{8}$$

For our first method of the above family we consider that:

$$a_0 = 0, \quad a_1 = -1, \quad a_2 = 2, \quad a_3 = -2, \quad a_4 = 1. \tag{9}$$

For the above method to require:

- the maximum algebraic order and
- four free parameters, in order the phase-lag and its first, second and third derivatives to be vanished.

The satisfaction of the above two requirement leads to the following relation:

$$b_0 = 5 - 2\,b_2 - 2\,b_1 - 2\,b_4 - 2\,b_3 \tag{10}$$

Now we apply the above method to the scalar test equation (4) and we get the following difference equation:

$$\sum_{i=0}^{4} A_i(\mathrm{v})\,\left(p_{n+i} + p_{n-i}\right) = 0 \tag{11}$$

where $\mathrm{v} = \omega h$, h is the step length and $A_i(\mathrm{v}),\ i = 0(1)4$ are polynomials of v.

The characteristic equation associated with (11) can be written as:

$$\sum_{i=0}^{4} A_i(\mathrm{v}) \left(\lambda^i + \lambda^{-i}\right) = 0 \tag{12}$$

where

$$\begin{aligned} A_0 &= \mathrm{v}^2 \ (5 - 2\,b_2 - 2\,b_1 - 2\,b_4 - 2\,b_3) \\ A_1 &= -1 + \mathrm{v}^2\, b_1 \\ A_2 &= 2 + \mathrm{v}^2\, b_2 \\ A_3 &= -2 + \mathrm{v}^2\, b_3 \\ A_4 &= 1 + \mathrm{v}^2\, b_4 \end{aligned} \tag{13}$$

We apply now the direct formula for the computation of the phase-lag (7) for $k = 4$ and for A_j, $j = 0, 1, \ldots, 4$ given by (13). This leads to the following equation:

$$\begin{aligned} phl &= \frac{T_0}{T_1} \\ T_0 &= 2 \left(1 + \mathrm{v}^2\, b_4\right) \cos(4\,\mathrm{v}) + 2 \left(-2 + \mathrm{v}^2\, b_3\right) \cos(3\,\mathrm{v}) \\ &\quad + 2 \left(2 + \mathrm{v}^2\, b_2\right) \cos(2\,\mathrm{v}) + 2 \left(-1 + \mathrm{v}^2\, b_1\right) \cos(\mathrm{v}) \\ &\quad + \mathrm{v}^2 \ (5 - 2\,b_2 - 2\,b_1 - 2\,b_4 - 2\,b_3) \\ T_1 &= 10 + 32\,\mathrm{v}^2\, b_4 + 18\,\mathrm{v}^2\, b_3 + 8\,\mathrm{v}^2\, b_2 + 2\,\mathrm{v}^2\, b_1 \end{aligned} \tag{14}$$

The phase-lag's first derivative is given by:

$$\begin{aligned} \dot{phl} &= \Big[4\,\mathrm{v}\, b_4 \cos(4\,\mathrm{v}) - 8 \left(1 + \mathrm{v}^2\, b_4\right) \sin(4\,\mathrm{v}) + 4\,\mathrm{v}\, b_3 \cos(3\,\mathrm{v}) \\ &\quad - 6 \left(-2 + \mathrm{v}^2\, b_3\right) \sin(3\,\mathrm{v}) + 4\,\mathrm{v}\, b_2 \cos(2\,\mathrm{v}) - 4 \left(2 + \mathrm{v}^2\, b_2\right) \sin(2\,\mathrm{v}) \\ &\quad + 4\,\mathrm{v}\, b_1 \cos(\mathrm{v}) - 2 \left(-1 + \mathrm{v}^2\, b_1\right) \sin(\mathrm{v}) + 2\,\mathrm{v} \ (5 - 2\,b_2 - 2\,b_1 - 2\,b_4 - 2\,b_3)\Big] \Big/ \\ &\quad \left(10 + 32\,\mathrm{v}^2\, b_4 + 18\,\mathrm{v}^2\, b_3 + 8\,\mathrm{v}^2\, b_2 + 2\,\mathrm{v}^2\, b_1\right) - \Big[2 \left(1 + \mathrm{v}^2\, b_4\right) \cos(4\,\mathrm{v}) \\ &\quad + 2 \left(-2 + \mathrm{v}^2\, b_3\right) \cos(3\,\mathrm{v}) + 2 \left(2 + \mathrm{v}^2\, b_2\right) \cos(2\,\mathrm{v}) + 2 \left(-1 + \mathrm{v}^2\, b_1\right) \cos(\mathrm{v}) \\ &\quad + \mathrm{v}^2 \left(5 - 2\,b_2 - 2\,b_1 - 2\,b_4 - 2\,b_3\right)\Big] (64\,\mathrm{v}\, b_4 + 36\,\mathrm{v}\, b_3 + 16\,\mathrm{v}\, b_2 + 4\,\mathrm{v}\, b_1) \Big/ \\ &\quad \left(10 + 32\,\mathrm{v}^2\, b_4 + 18\,\mathrm{v}^2\, b_3 + 8\,\mathrm{v}^2\, b_2 + 2\,\mathrm{v}^2\, b_1\right)^2 \end{aligned} \tag{15}$$

The phase-lag's second derivative is given by:

$$\ddot{phl} = \Big[4\, b_4 \cos(4\,\mathrm{v}) - 32\,\mathrm{v}\, b_4 \sin(4\,\mathrm{v}) - 32 \left(1 + \mathrm{v}^2\, b_4\right) \cos(4\,\mathrm{v})$$

$$
\begin{aligned}
&+ 4\, b_3 \cos(3\,\mathrm{v}) - 24\,\mathrm{v}\, b_3 \sin(3\,\mathrm{v}) - 18\,\left(-2 + \mathrm{v}^2\, b_3\right)\,\cos(3\,\mathrm{v})\\
&+ 4\, b_2 \cos(2\,\mathrm{v}) - 16\,\mathrm{v}\, b_2 \sin(2\,\mathrm{v}) - 8\,\left(2 + \mathrm{v}^2\, b_2\right)\,\cos(2\,\mathrm{v})\\
&+ 4\, b_1 \cos(\mathrm{v}) - 8\,\mathrm{v}\, b_1 \sin(\mathrm{v}) - 2\,\left(-1 + \mathrm{v}^2\, b_1\right)\,\cos(\mathrm{v})\\
&+\; 10 - 4\, b_2 -\; 4\, b_1 - 4\, b_4 - 4\, b_3]\,/(\mathrm{T}_2) - 2\Big[4\,\mathrm{v}\, b_4 \cos(4\,\mathrm{v})\\
&- 8\,\left(1 + \mathrm{v}^2\, b_4\right)\,\sin(4\,\mathrm{v}) + 4\,\mathrm{v}\, b_3 \cos(3\,\mathrm{v}) - 6\,\left(-2 + \mathrm{v}^2\, b_3\right)\,\sin(3\,\mathrm{v})\\
&+ 4\,\mathrm{v}\, b_2 \cos(2\,\mathrm{v}) - 4\,\left(2 + \mathrm{v}^2\, b_2\right)\,\sin(2\,\mathrm{v}) + 4\,\mathrm{v}\, b_1 \cos(\mathrm{v})\\
&- 2\,\left(-1 + \mathrm{v}^2\, b_1\right)\,\sin(\mathrm{v}) + 2\,\mathrm{v}\,\mathrm{T}_3\Big]\\
&(64\,\mathrm{v}\, b_4 + 36\,\mathrm{v}\, b_3 + 16\,\mathrm{v}\, b_2 + 4\,\mathrm{v}\, b_1)\,/{\mathrm{T}_2}^2 +\\
&2\Big[2\,\left(1 + \mathrm{v}^2\, b_4\right)\,\cos(4\,\mathrm{v}) + 2\,\left(-2 + \mathrm{v}^2\, b_3\right)\,\cos(3\,\mathrm{v})\\
&+ 2\,\left(2 + \mathrm{v}^2\, b_2\right)\,\cos(2\,\mathrm{v}) + 2\,\left(-1 + \mathrm{v}^2\, b_1\right)\,\cos(\mathrm{v})\\
&+ \mathrm{v}^2\,\mathrm{T}_3\Big]\,(64\,\mathrm{v}\, b_4 + 36\,\mathrm{v}\, b_3 + 16\,\mathrm{v}\, b_2 + 4\,\mathrm{v}\, b_1)^2\,/{\mathrm{T}_2}^3\\
&-\,\Big[2\,\left(1 + \mathrm{v}^2\, b_4\right)\,\cos(4\,\mathrm{v}) + 2\,\left(-2 + \mathrm{v}^2\, b_3\right)\,\cos(3\,\mathrm{v})\\
&+ 2\,\left(2 + \mathrm{v}^2\, b_2\right)\,\cos(2\,\mathrm{v}) + 2\,\left(-1 + \mathrm{v}^2\, b_1\right)\,\cos(\mathrm{v})\\
&+ \mathrm{v}^2\,\mathrm{T}_3\Big]\,(64\, b_4 + 36\, b_3 + 16\, b_2 + 4\, b_1)\,/{\mathrm{T}_2}^2
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{T}_2 &= 10 + 32\,\mathrm{v}^2\, b_4 + 18\,\mathrm{v}^2\, b_3 + 8\,\mathrm{v}^2\, b_2 + 2\,\mathrm{v}^2\, b_1\\
\mathrm{T}_3 &= 5 - 2\, b_2 - 2\, b_1 - 2\, b_4 - 2\, b_3
\end{aligned}
\tag{16}
$$

The phase-lag's third derivative is given by:

$$
\begin{aligned}
p\hat{h}l = \Big[&-48\, b_4 \sin(4\,\mathrm{v}) - 192\,\mathrm{v}\, b_4 \cos(4\,\mathrm{v})\\
&+ 128\,\left(1 + \mathrm{v}^2\, b_4\right)\,\sin(4\,\mathrm{v}) - 36\, b_3 \sin(3\,\mathrm{v})\\
&- 108\,\mathrm{v}\, b_3 \cos(3\,\mathrm{v}) + 54\,\left(-2 + \mathrm{v}^2\, b_3\right)\,\sin(3\,\mathrm{v}) - 24\, b_2 \sin(2\,\mathrm{v})\\
&- 48\,\mathrm{v}\, b_2 \cos(2\,\mathrm{v}) + 16\,\left(2 + \mathrm{v}^2\, b_2\right)\,\sin(2\,\mathrm{v})\\
&- 12\, b_1 \sin(\mathrm{v}) - 12\,\mathrm{v}\, b_1 \cos(\mathrm{v})\\
&+ 2\,\left(-1 + \mathrm{v}^2\, b_1\right)\,\sin(\mathrm{v})\Big]\,/(\mathrm{T}_5) - 3\Big[4\, b_4 \cos(4\,\mathrm{v}) - 32\,\mathrm{v}\, b_4 \sin(4\,\mathrm{v})\\
&- 32\,\left(1 + \mathrm{v}^2\, b_4\right)\,\cos(4\,\mathrm{v}) + 4\, b_3 \cos(3\,\mathrm{v}) - 24\,\mathrm{v}\, b_3 \sin(3\,\mathrm{v})\\
&- 18\,\left(-2 + \mathrm{v}^2\, b_3\right)\,\cos(3\,\mathrm{v}) + 4\, b_2 \cos(2\,\mathrm{v}) - 16\,\mathrm{v}\, b_2 \sin(2\,\mathrm{v})
\end{aligned}
$$

$$
\begin{aligned}
&- 8 \left(2 + v^2 b_2\right) \cos(2 v) + 4 b_1 \cos(v) \\
&- 8 v b_1 \sin(v) - 2 \left(-1 + v^2 b_1\right) \cos(v) \\
&+ 10 - 4 b_2 - 4 b_1 - 4 b_4 - 4 b_3] T_4/{T_5}^2 + 6 \Big[4 v b_4 \cos(4 v) \\
&- 8 \left(1 + v^2 b_4\right) \sin(4 v) + 4 v b_3 \cos(3 v) \\
&- 6 \left(-2 + v^2 b_3\right) \sin(3 v) + 4 v b_2 \cos(2 v) \\
&- 4 \left(2 + v^2 b_2\right) \sin(2 v) + 4 v b_1 \cos(v) \\
&- 2 \left(-1 + v^2 b_1\right) \sin(v) + 2 v T_6\Big] {T_4}^2/{T_5}^3 - 3 \Big[4 v b_4 \cos(4 v) \\
&- 8 \left(1 + v^2 b_4\right) \sin(4 v) + 4 v b_3 \cos(3 v) - 6 \left(-2 + v^2 b_3\right) \sin(3 v) \\
&+ 4 v b_2 \cos(2 v) - 4 \left(2 + v^2 b_2\right) \sin(2 v) + 4 v b_1 \cos(v) \\
&- 2 \left(-1 + v^2 b_1\right) \sin(v) + 2 v T_6\Big] (64 b_4 + 36 b_3 + 16 b_2 + 4 b_1)/{T_5}^2 \\
&- 6 \Big[2 (1 + v^2 b_4) \cos(4 v) + 2 \left(-2 + v^2 b_3\right) \cos(3 v) \\
&+ 2 \left(2 + v^2 b_2\right) \cos(2 v) + 2 \left(-1 + v^2 b_1\right) \cos(v) \\
&+ v^2 T_6\Big] {T_4}^3/{T_5}^4 + 6 \Big[2 \left(1 + v^2 b_4\right) \cos(4 v) \\
&+ 2 \left(-2 + v^2 b_3\right) \cos(3 v) + 2 \left(2 + v^2 b_2\right) \cos(2 v) \\
&+ 2 \left(-1 + v^2 b_1\right) \cos(v) + v^2 T_6\Big] T_4 \\
&(64 b_4 + 36 b_3 + 16 b_2 + 4 b_1) \big/ {T_5}^3 \\
T_4 &= 64 v b_4 + 36 v b_3 + 16 v b_2 + 4 v b_1 \\
T_5 &= 10 + 32 v^2 b_4 + 18 v^2 b_3 + 8 v^2 b_2 + 2 v^2 b_1 \\
T_6 &= 5 - 2 b_2 - 2 b_1 - 2 b_4 - 2 b_3
\end{aligned} \tag{17}
$$

We demand that the phase-lag and its first, second and third derivatives to be equal to zero, i.e. we demand the satisfaction of the relations (14), (15), (16) and (17). Based on the above we obtain the coefficients mentioned in the Appendix A.

The behavior of the coefficients is given in Fig. 1.

The local truncation error of the new proposed method is given by:

$$
LTE = -\frac{58061 h^{12}}{31933440} \left(y_n^{(12)} + 4 \omega^2 \; y_n^{(10)} + 6 \omega^4 \; y_n^{(8)} + 4 \omega^6 \; y_n^{(6)} + \omega^8 \; y_n^{(4)} \right) \tag{18}
$$

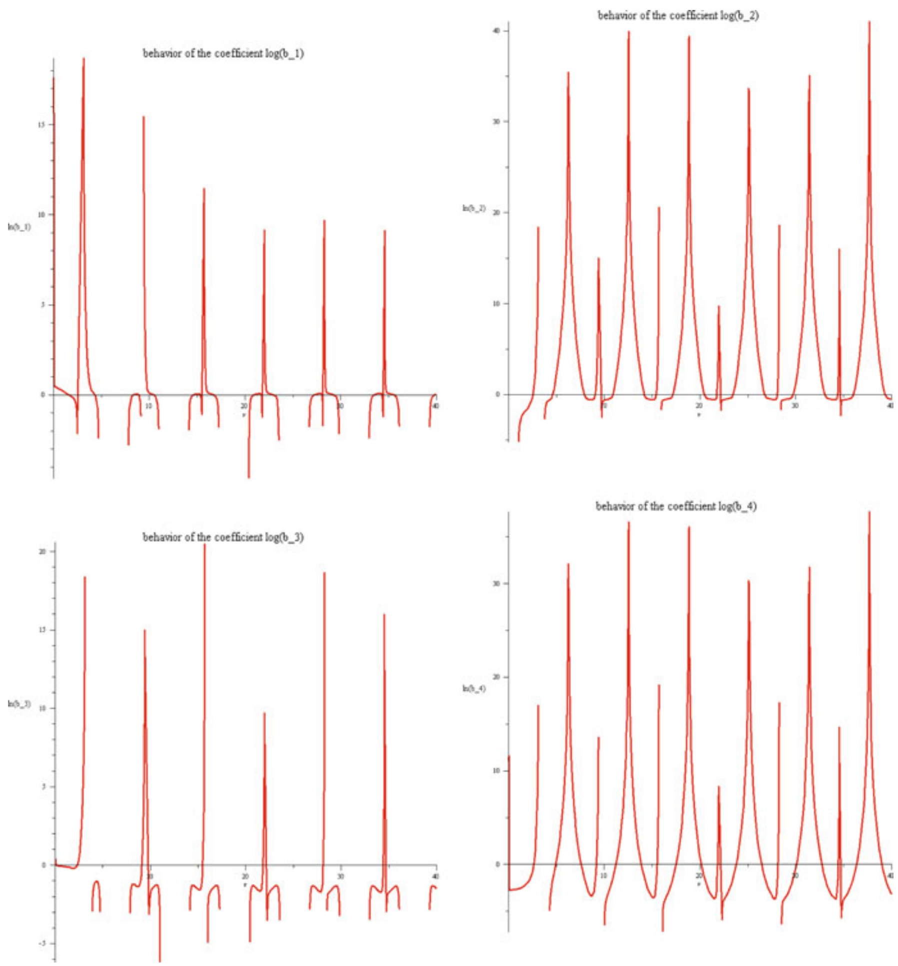

Fig. 1 Behavior of the coefficients of the new proposed method given by (56)–(59) for several values of v

3.2 Second method of the family with vanished phase-lag and its first, second and third derivatives

Consider the family of methods (8) to integrate $q'' = f(x, q)$. For this method we have the same requirements as above.

Now we consider as free parameters the coefficients $a_i,\ i = 0(1)3$ with $a_4 = 1$ and

$$\begin{aligned} b_0 &= \frac{17273}{72576}, \quad b_1 = \frac{280997}{181440}, \quad b_2 = -\frac{33961}{181440}, \\ b_3 &= \frac{173531}{181440}, \quad b_4 = \frac{45767}{725760} \end{aligned} \tag{19}$$

We apply the above method to the scalar test equation (4) and we get the difference equation (11). The associated characteristic equation is given by (12) with:

$$\begin{aligned} A_0 &= a_0 + \frac{17273}{72576}\,\mathrm{v}^2 \\ A_1 &= a_1 + \frac{280997}{181440}\,\mathrm{v}^2 \\ A_2 &= a_2 - \frac{33961}{181440}\,\mathrm{v}^2 \\ A_3 &= a_3 + \frac{173531}{181440}\,\mathrm{v}^2 \\ A_4 &= 1 + \frac{45767}{725760}\,\mathrm{v}^2 \end{aligned} \tag{20}$$

We apply now the direct formula for the computation of the phase-lag (7) for $k = 4$ and for A_j, $j = 0, 1, \ldots, 4$ given by (20). This leads to the following equation:

$$\begin{aligned} phl &= \frac{T_7}{T_8} \\ T_7 &= 2\left(1 + \frac{45767}{725760}\,\mathrm{v}^2\right)\cos(4\,\mathrm{v}) + 2\left(a_3 + \frac{173531}{181440}\,\mathrm{v}^2\right)\cos(3\,\mathrm{v}) \\ &\quad + 2\left(a_2 - \frac{33961}{181440}\,\mathrm{v}^2\right)\cos(2\,\mathrm{v}) + 2\left(a_1 + \frac{280997}{181440}\,\mathrm{v}^2\right)\cos(\mathrm{v}) \\ &\quad + a_0 + \frac{17273}{72576}\,\mathrm{v}^2 \\ T_8 &= 32 + \frac{125}{6}\,H^2 + 18\,a_3 + 8\,a_2 + 2\,a_1 \end{aligned} \tag{21}$$

The phase-lag's first derivative is given by:

$$\begin{aligned} \dot{phl} = &\left[\frac{45767}{181440}\,\mathrm{v}\cos(4\,\mathrm{v}) - 8\left(1 + \frac{45767}{725760}\,\mathrm{v}^2\right)\sin(4\,\mathrm{v}) + \frac{173531}{45360}\,\mathrm{v}\cos(3\,\mathrm{v})\right. \\ &- 6\left(a_3 + \frac{173531}{181440}\,\mathrm{v}^2\right)\sin(3\,\mathrm{v}) - \frac{33961}{45360}\,\mathrm{v}\cos(2\,\mathrm{v}) \\ &- 4\left(a_2 - \frac{33961}{181440}\,\mathrm{v}^2\right)\sin(2\,\mathrm{v}) + \frac{280997}{45360}\,\mathrm{v}\cos(\mathrm{v}) \\ &\left.- 2\left(a_1 + \frac{280997}{181440}\,\mathrm{v}^2\right)\sin(\mathrm{v}) + \frac{17273}{36288}\,\mathrm{v}\right]\left(32 + \frac{125}{6}\,\mathrm{v}^2\right. \\ &\left.+ 18\,a_3 + 8\,a_2 + 2\,a_1\right)^{-1} - \frac{125}{3}\left[2\left(1 + \frac{45767}{725760}\,\mathrm{v}^2\right)\cos(4\,\mathrm{v})\right. \\ &+ 2\left(a_3 + \frac{173531}{181440}\,\mathrm{v}^2\right)\cos(3\,\mathrm{v}) \end{aligned}$$

$$+2\left(a_2-\frac{33961}{181440}\,v^2\right)\cos(2\,v)+2\left(a_1+\frac{280997}{181440}\,v^2\right)\cos(v)$$
$$\left.+a_0+\frac{17273}{72576}\,v^2\right]v$$
$$\left(32+\frac{125}{6}\,v^2+18\,a_3+8\,a_2+2\,a_1\right)^{-2} \tag{22}$$

The phase-lag's second derivative is given by:

$$\ddot{phl}=\left[\frac{45767}{181440}\cos(4\,v)-\frac{45767}{22680}\,v\sin(4\,v)\right.$$
$$-32\left(1+\frac{45767}{725760}\,v^2\right)\cos(4\,v)+\frac{173531}{45360}\cos(3\,v)-\frac{173531}{7560}\,v\sin(3\,v)$$
$$-18\left(a_3+\frac{173531}{181440}\,v^2\right)\cos(3\,v)-\frac{33961}{45360}\cos(2\,v)+\frac{33961}{11340}\,v\sin(2\,v)$$
$$-8\left(a_2-\frac{33961}{181440}\,v^2\right)\cos(2\,v)+\frac{280997}{45360}\cos(v)-\frac{280997}{22680}\,v\sin(v)$$
$$\left.-2\left(a_1+\frac{280997}{181440}\,v^2\right)\cos(v)+\frac{17273}{36288}\right]\left(32+\frac{125}{6}\,v^2\right.$$
$$\left.+18\,a_3+8\,a_2+2\,a_1\right)^{-1}-\frac{250}{3}\left[\frac{45767}{181440}\,v\cos(4\,v)\right.$$
$$-8\left(1+\frac{45767}{725760}\,v^2\right)\sin(4\,v)+\frac{173531}{45360}\,v\cos(3\,v)$$
$$-6\left(a_3+\frac{173531}{181440}\,v^2\right)\sin(3\,v)-\frac{33961}{45360}\,v\cos(2\,v)$$
$$-4\left(a_2-\frac{33961}{181440}\,v^2\right)\sin(2\,v)+\frac{280997}{45360}\,v\cos(v)$$
$$\left.-2\left(a_1+\frac{280997}{181440}\,v^2\right)\sin(v)+\frac{17273}{36288}\,v\right]v\left(32+\frac{125}{6}\,v^2+18\,a_3+8\,a_2\right.$$
$$\left.+2\,a_1\right)^{-2}+\frac{31250}{9}\left[2\left(1+\frac{45767}{725760}\,v^2\right)\cos(4\,v)\right.$$
$$+2\left(a_3+\frac{173531}{181440}\,v^2\right)\cos(3\,v)+2\left(a_2-\frac{33961}{181440}\,v^2\right)$$
$$\left.\times\cos(2\,v)+2\left(a_1+\frac{280997}{181440}\,v^2\right)\cos(v)+a_0+\frac{17273}{72576}\,v^2\right]$$
$$v^2\left(32+\frac{125}{6}\,v^2+18\,a_3+8\,a_2+2\,a_1\right)^{-3}$$

$$-\frac{125}{3}\left[2\left(1+\frac{45767}{725760}\,v^2\right)\cos(4\,v)+2\left(a_3+\frac{173531}{181440}\,v^2\right)\cos(3\,v)\right.$$
$$\left.+2\left(a_2-\frac{33961}{181440}\,v^2\right)\cos(2\,v)+2\left(a_1+\frac{280997}{181440}\,v^2\right)\cos(v)+a_0+\frac{17273}{72576}\,v^2\right]$$
$$\left(32+\frac{125}{6}\,v^2+18\,a_3+8\,a_2+2\,a_1\right)^{-2} \tag{23}$$

The phase-lag's third derivative is given by:

$$\hat{phl}=\left[-\frac{45767}{15120}\sin(4\,v)-\frac{45767}{3780}\,v\cos(4\,v)\right.$$
$$+128\left(1+\frac{45767}{725760}\,v^2\right)\sin(4\,v)-\frac{173531}{5040}\sin(3\,v)-\frac{173531}{1680}\,v\cos(3\,v)$$
$$+54\left(a_3+\frac{173531}{181440}\,v^2\right)\sin(3\,v)+\frac{33961}{7560}\sin(2\,v)+\frac{33961}{3780}\,v\cos(2\,v)$$
$$+16\left(a_2-\frac{33961}{181440}\,v^2\right)\sin(2\,v)-\frac{280997}{15120}\sin(v)-\frac{280997}{15120}\,v\cos(v)$$
$$\left.+2\left(a_1+\frac{280997}{181440}\,v^2\right)\sin(v)\right]\left(32+\frac{125}{6}\,v^2+18\,a_3+8\,a_2+2\,a_1\right)^{-1}$$
$$-125\left[\frac{45767}{181440}\cos(4\,v)-\frac{45767}{22680}\,v\sin(4\,v)\right.$$
$$-32\left(1+\frac{45767}{725760}\,v^2\right)\cos(4\,v)+\frac{173531}{45360}\cos(3\,v)-\frac{173531}{7560}\,v\sin(3\,v)$$
$$-18\left(a_3+\frac{173531}{181440}\,v^2\right)\cos(3\,v)-\frac{33961}{45360}\cos(2\,v)+\frac{33961}{11340}\,v\sin(2\,v)$$
$$-8\left(a_2-\frac{33961}{181440}\,v^2\right)\cos(2\,v)+\frac{280997}{45360}\cos(v)-\frac{280997}{22680}\,v\sin(v)$$
$$\left.-2\left(a_1+\frac{280997}{181440}\,v^2\right)\cos(v)+\frac{17273}{36288}\right]v\left(32+\frac{125}{6}\,v^2+18\,a_3+8\,a_2+2\,a_1\right)^{-2}$$
$$+\frac{31250}{3}\left[\frac{45767}{181440}\,v\cos(4\,v)-8\left(1+\frac{45767}{725760}\,v^2\right)\sin(4\,v)+\frac{173531}{45360}\,v\cos(3\,v)\right.$$
$$-6\left(a_3+\frac{173531}{181440}\,v^2\right)\sin(3\,v)-\frac{33961}{45360}\,v\cos(2\,v)$$
$$-4\left(a_2-\frac{33961}{181440}\,v^2\right)\sin(2\,v)+\frac{280997}{45360}\,v\cos(v)$$
$$\left.-2\left(a_1+\frac{280997}{181440}\,v^2\right)\sin(v)+\frac{17273}{36288}\,v\right]$$

$$
\mathrm{v}^2\left(32+\frac{125}{6}\,\mathrm{v}^2+18\,a_3+8\,a_2+\,2\,a_1\right)^{-3}-125\left[\frac{45767}{181440}\,\mathrm{v}\cos(4\,\mathrm{v})\right.
-8\left(1+\frac{45767}{725760}\,\mathrm{v}^2\right)\sin(4\,\mathrm{v})+\frac{173531}{45360}\,\mathrm{v}\cos(3\,\mathrm{v})
-6\left(a_3+\frac{173531}{181440}\,\mathrm{v}^2\right)\sin(3\,\mathrm{v})-\frac{33961}{45360}\,\mathrm{v}\cos(2\,\mathrm{v})
-4\left(a_2-\frac{33961}{181440}\,\mathrm{v}^2\right)\sin(2\,\mathrm{v})+\frac{280997}{45360}\,\mathrm{v}\cos(\mathrm{v})
\left.-2\left(a_1+\frac{280997}{181440}\,\mathrm{v}^2\right)\sin(\mathrm{v})+\frac{17273}{36288}\,\mathrm{v}\right]\left(32+\frac{125}{6}\,\mathrm{v}^2\right.
\left.+18\,a_3+8\,a_2+2\,a_1\right)^{-2}-\frac{3906250}{9}\left[2\left(1+\frac{45767}{725760}\,\mathrm{v}^2\right)\cos(4\,\mathrm{v})\right.
+2\left(a_3+\frac{173531}{181440}\,\mathrm{v}^2\right)\cos(3\,\mathrm{v})+2\left(a_2-\frac{33961}{181440}\,\mathrm{v}^2\right)\cos(2\,\mathrm{v})
\left.+2\left(a_1+\frac{280997}{181440}\,\mathrm{v}^2\right)\cos(\mathrm{v})+a_0+\frac{17273}{72576}\,\mathrm{v}^2\right]
\mathrm{v}^3\left(32+\frac{125}{6}\,\mathrm{v}^2+18\,a_3+8\,a_2+2\,a_1\right)^{-4}
+\frac{31250}{3}\left[2\left(1+\frac{45767}{725760}\,\mathrm{v}^2\right)\cos(4\,\mathrm{v})+2\left(a_3+\frac{173531}{181440}\,\mathrm{v}^2\right)\cos(3\,\mathrm{v})\right.
\left.+2\left(a_2-\frac{33961}{181440}\,\mathrm{v}^2\right)\cos\left(2\,\mathrm{v}\right)+2\left(a_1+\frac{280997}{181440}\,\mathrm{v}^2\right)\cos(\mathrm{v})+a_0+\frac{17273}{72576}\,\mathrm{v}^2\right]
\mathrm{v}\left(32+\frac{125}{6}\,\mathrm{v}^2+18\,a_3+8\,a_2+2\,a_1\right)^{-3} \tag{24}
$$

We demand that the phase-lag and its first, second and third derivatives to be equal to zero, i.e. we demand the satisfaction of the relations (21), (22), (23) and (24). Based on the above we obtain the coefficients mentioned in the Appendix B.

The behavior of the coefficients is given in Fig. 2.

The local truncation error of the new proposed method is given by:

$$
\mathrm{LTE}=-\frac{58061\,h^{12}}{31933440}\left(y_n^{(12)}+20\,\omega^6\;y_n^{(6)}+45\,\omega^8\;y_n^{(4)}+36\,\omega^{10}\;y_n^{(2)}+10\,\omega^{12}\;y_n\right) \tag{25}
$$

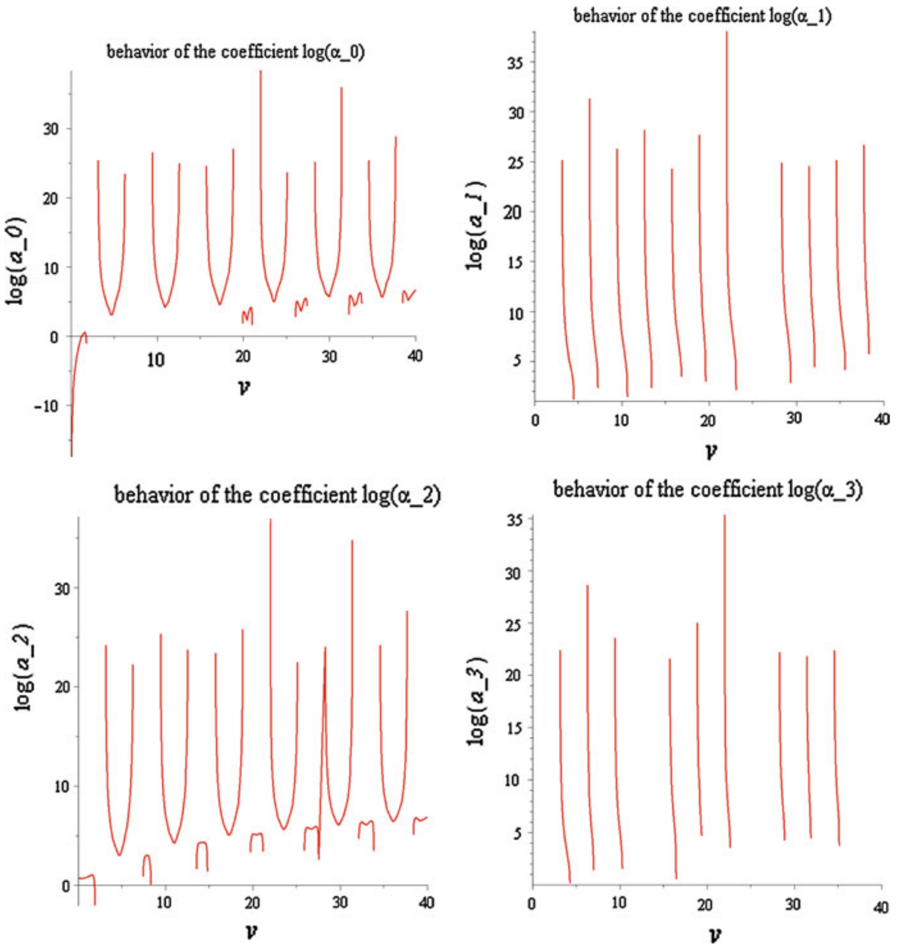

Fig. 2 Behavior of the coefficients of the new proposed method given by (61)–(64) for several values of v

3.3 Third method of the family with vanished phase-lag and its first, second and third derivatives

Consider again the family of methods (8) to integrate $q'' = f(x, q)$.

For this method we have the same requirements as above. Now we consider as free parameters the coefficients $a_i,\ i = 1(1)4$ with $a_0 = 0$ and

$$\begin{aligned} b_0 &= \frac{17273}{72576}, \quad b_1 = \frac{280997}{181440}, \quad b_2 = -\frac{33961}{181440}, \\ b_3 &= \frac{173531}{181440}, \quad b_4 = \frac{45767}{725760} \end{aligned} \tag{26}$$

We apply the above method to the scalar test equation (4) and we get the difference equation (11). The associated characteristic equation is given by (12) with:

$$\begin{aligned} A_0 &= \frac{17273}{72576}\,\mathrm{v}^2 \\ A_1 &= a_1 + \frac{280997}{181440}\,H^2 \\ A_2 &= a_2 - \frac{33961}{181440}\,\mathrm{v}^2 \\ A_3 &= a_3 + \frac{173531}{181440}\,\mathrm{v}^2 \\ A_4 &= a_4 + \frac{45767}{725760}\,\mathrm{v}^2 \end{aligned} \tag{27}$$

We apply now the direct formula for the computation of the phase-lag (7) for $k = 4$ and for $A_j,\ j = 0,\ 1,\ \ldots, 4$ given by (27). This leads to the following equation:

$$\begin{aligned} phl &= \frac{T_{13}}{T_{14}} \\ T_{13} &= 2\left(a_4 + \frac{45767}{725760}\,\mathrm{v}^2\right)\cos(4\,\mathrm{v}) \\ &\quad + 2\left(a_3 + \frac{173531}{181440}\,\mathrm{v}^2\right)\cos(3\,\mathrm{v}) + 2\left(a_2 - \frac{33961}{181440}\,\mathrm{v}^2\right)\cos(2\,\mathrm{v}) \\ &\quad + 2\left(a_1 + \frac{280997}{181440}\,\mathrm{v}^2\right)\cos(\mathrm{v}) + \frac{17273}{72576}\,\mathrm{v}^2 \\ T_{14} &= 32\,a_4 + \frac{125}{6}\,H^2 + 18\,a_3 + 8\,a_2 + 2\,a_1 \end{aligned} \tag{28}$$

The phase-lag's first derivative is given by:

$$\begin{aligned} p\dot{h}l &= \left[\frac{45767}{181440}\,\mathrm{v}\cos(4\,\mathrm{v}) - 8\left(a_4 + \frac{45767}{725760}\,\mathrm{v}^2\right)\sin(4\,\mathrm{v})\right. \\ &\quad + \frac{173531}{45360}\,\mathrm{v}\cos(3\,\mathrm{v}) - 6\left(a_3 + \frac{173531}{181440}\,\mathrm{v}^2\right)\sin(3\,\mathrm{v}) \\ &\quad - \frac{33961}{45360}\,\mathrm{v}\cos(2\,\mathrm{v}) - 4\left(a_2 - \frac{33961}{181440}\,\mathrm{v}^2\right)\sin(2\,\mathrm{v}) + \frac{280997}{45360}\,\mathrm{v}\cos(\mathrm{v}) \\ &\quad \left.- 2\left(a_1 + \frac{280997}{181440}\,\mathrm{v}^2\right)\sin(\mathrm{v}) + \frac{17273}{36288}\,\mathrm{v}\right]\left(32\,a_4 + \frac{125}{6}\,\mathrm{v}^2\right. \\ &\quad \left.+ 18\,a_3 + 8\,a_2 + 2\,a_1\right)^{-1} - \frac{125}{3}\left[2\left(a_4 + \frac{45767}{725760}\,\mathrm{v}^2\right)\cos(4\,\mathrm{v})\right. \\ &\quad + 2\left(a_3 + \frac{173531}{181440}\,\mathrm{v}^2\right)\cos(3\,\mathrm{v}) + 2\left(a_2 - \frac{33961}{181440}\,\mathrm{v}^2\right)\cos(2\,\mathrm{v}) \end{aligned}$$

$$+2\left(a_1+\frac{280997}{181440}\,v^2\right)\cos(v)+\frac{17273}{72576}\,v^2\Bigg]$$

$$v\left(32\,a_4+\frac{125}{6}\,v^2+18\,a_3+8\,a_2+2\,a_1\right)^{-2} \tag{29}$$

The phase-lag's second derivative is given by:

$$\begin{aligned}
\ddot{phl} = & \Bigg[\frac{45767}{181440}\cos(4\,v)-\frac{45767}{22680}\,v\sin(4\,v) \\
& -32\left(a_4+\frac{45767}{725760}\,v^2\right)\cos(4\,v)+\frac{173531}{45360}\cos(3\,v)-\frac{173531}{7560}\,v\sin(3\,v) \\
& -18\left(a_3+\frac{173531}{181440}\,v^2\right)\cos(3\,v)-\frac{33961}{45360}\cos(2\,v)+\frac{33961}{11340}\,v\sin(2\,v) \\
& -8\left(a_2-\frac{33961}{181440}\,v^2\right)\cos(2\,v)+\frac{280997}{45360}\cos(v)-\frac{280997}{22680}\,v\sin(v) \\
& -2\left(a_1+\frac{280997}{181440}\,v^2\right)\cos(v)+\frac{17273}{36288}\Bigg]\Bigg(32\,a_4+\frac{125}{6}\,v^2 \\
& +18\,a_3+8\,a_2+2\,a_1\Bigg)^{-1}-\frac{250}{3}\Bigg[\frac{45767}{181440}\,v\cos(4\,v) \\
& -8\left(a_4+\frac{45767}{725760}\,v^2\right)\sin(4\,v)+\frac{173531}{45360}\,v\cos(3\,v) \\
& -6\left(a_3+\frac{173531}{181440}\,v^2\right)\sin(3\,v)-\frac{33961}{45360}\,v\cos(2\,v) \\
& -4\left(a_2-\frac{33961}{181440}\,v^2\right)\sin(2\,v)+\frac{280997}{45360}\,v\cos(v) \\
& -2\left(a_1+\frac{280997}{181440}\,v^2\right)\sin(v)+\frac{17273}{36288}\,v\Bigg]v\Bigg(32\,a_4+\frac{125}{6}\,v^2+18\,a_3+8\,a_2 \\
& +2\,a_1\Bigg)^{-2}+\frac{31250}{9}\Bigg[2\left(a_4+\frac{45767}{725760}\,v^2\right)\cos(4\,v) \\
& +2\left(a_3+\frac{173531}{181440}\,v^2\right)\cos(3\,v)+2\left(a_2-\frac{33961}{181440}\,v^2\right) \\
& \times\cos(2\,v)+2\left(a_1+\frac{280997}{181440}\,v^2\right)\cos(v)+\frac{17273}{72576}\,v^2\Bigg] \\
& v^2\left(32\,a_4+\frac{125}{6}\,v^2+18\,a_3+8\,a_2+2\,a_1\right)^{-3} \\
& -\frac{125}{3}\Bigg[2\left(a_4+\frac{45767}{725760}\,v^2\right)\cos(4\,v)+2\left(a_3+\frac{173531}{181440}\,v^2\right)\cos(3\,v) \\
& +2\left(a_2-\frac{33961}{181440}\,v^2\right)\cos(2\,v)+2\left(a_1+\frac{280997}{181440}\,v^2\right)\cos(v)+\frac{17273}{72576}\,v^2\Bigg]
\end{aligned}$$

$$\left(32\,a_4 + \frac{125}{6}\,v^2 + 18\,a_3 + 8\,a_2 + 2\,a_1\right)^{-2} \tag{30}$$

The phase-lag's third derivative is given by:

$$\begin{aligned}
p\hat{h}l = &\left[-\frac{45767}{15120}\,\sin(4\,v) - \frac{45767}{3780}\,v\cos(4\,v)\right.\\
&+128\left(a_4 + \frac{45767}{725760}\,v^2\right)\sin(4\,v) - \frac{173531}{5040}\,\sin(3\,v) - \frac{173531}{1680}\,v\cos(3\,v)\\
&+54\left(a_3 + \frac{173531}{181440}\,v^2\right)\sin(3\,v) + \frac{33961}{7560}\,\sin(2\,v) + \frac{33961}{3780}\,v\cos(2\,v)\\
&+16\left(a_2 - \frac{33961}{181440}\,v^2\right)\sin(2\,v) - \frac{280997}{15120}\,\sin(v) - \frac{280997}{15120}\,v\cos(v)\\
&\left.+2\left(a_1 + \frac{280997}{181440}\,v^2\right)\sin(v)\right]\left(32\,a_4 + \frac{125}{6}\,v^2 + 18\,a_3 + 8\,a_2\right.\\
&\left.+2\,a_1\right)^{-1} - 125\left[\frac{45767}{181440}\,\cos(4\,v) - \frac{45767}{22680}\,v\sin(4\,v)\right.\\
&-32\left(a_4 + \frac{45767}{725760}\,v^2\right)\cos(4\,v) + \frac{173531}{45360}\,\cos(3\,v) - \frac{173531}{7560}\,v\sin(3\,v)\\
&-18\left(a_3 + \frac{173531}{181440}\,v^2\right)\cos(3\,v) - \frac{33961}{45360}\,\cos(2\,v) + \frac{33961}{11340}\,v\sin(2\,v)\\
&-8\left(a_2 - \frac{33961}{181440}\,v^2\right)\cos(2\,v) + \frac{280997}{45360}\,\cos(v) - \frac{280997}{22680}\,v\sin(v)\\
&\left.-2\left(a_1 + \frac{280997}{181440}\,v^2\right)\cos(v) + \frac{17273}{36288}\right]\\
&v\left(32\,a_4 + \frac{125}{6}\,v^2 + 18\,a_3 + 8\,a_2 + 2\,a_1\right)^{-2}\\
&+\frac{31250}{3}\left[\frac{45767}{181440}\,v\cos(4\,v) - 8\left(a_4 + \frac{45767}{725760}\,v^2\right)\sin(4\,v)\right.\\
&+\frac{173531}{45360}\,v\cos(3\,v) - 6\left(a_3 + \frac{173531}{181440}\,v^2\right)\sin(3\,v) - \frac{33961}{45360}\,v\cos(2\,v)\\
&-4\left(a_2 - \frac{33961}{181440}\,v^2\right)\sin(2\,v) + \frac{280997}{45360}\,v\cos(v)\\
&\left.-2\left(a_1 + \frac{280997}{181440}\,v^2\right)\sin(v) + \frac{17273}{36288}\,v\right]v^2\left(32\,a_4 + \frac{125}{6}\,v^2 + 18\,a_3\right.\\
&\left.+8\,a_2 + 2\,a_1\right)^{-3} - 125\left[\frac{45767}{181440}\,v\cos(4\,v)\right.
\end{aligned}$$

$$
\begin{aligned}
&-8\left(a_4+\frac{45767}{725760}\,\mathrm{v}^2\right)\sin(4\,\mathrm{v})+\frac{173531}{45360}\,\mathrm{v}\cos(3\,\mathrm{v})\\
&-6\left(a_3+\frac{173531}{181440}\,\mathrm{v}^2\right)\sin(3\,\mathrm{v})-\frac{33961}{45360}\,\mathrm{v}\cos(2\,\mathrm{v})\\
&-4\left(a_2-\frac{33961}{181440}\,\mathrm{v}^2\right)\sin(2\,\mathrm{v})+\frac{280997}{45360}\,\mathrm{v}\cos(\mathrm{v})\\
&-2\left(a_1+\frac{280997}{181440}\,\mathrm{v}^2\right)\sin(\mathrm{v})+\frac{17273}{36288}\,\mathrm{v}\Bigg]\Bigg(32\,a_4+\frac{125}{6}\,\mathrm{v}^2\\
&+18\,a_3+8\,a_2+2\,a_1\Bigg)^{-2}-\frac{3906250}{9}\Bigg[2\left(a_4+\frac{45767}{725760}\,\mathrm{v}^2\right)\cos(4\,\mathrm{v})\\
&+2\left(a_3+\frac{173531}{181440}\,\mathrm{v}^2\right)\cos(3\,\mathrm{v})+2\left(a_2-\frac{33961}{181440}\,\mathrm{v}^2\right)\\
&\cos(2\,\mathrm{v})+2\left(a_1+\frac{280997}{181440}\,\mathrm{v}^2\right)\cos(\mathrm{v})+\frac{17273}{72576}\,\mathrm{v}^2\Bigg]\\
&\mathrm{v}^3\left(32\,a_4+\frac{125}{6}\,\mathrm{v}^2+18\,a_3+8\,a_2+2\,a_1\right)^{-4}\\
&+\frac{31250}{3}\Bigg[2\left(a_4+\frac{45767}{725760}\,\mathrm{v}^2\right)\cos(4\,\mathrm{v})+2\left(a_3+\frac{173531}{181440}\,\mathrm{v}^2\right)\cos(3\,\mathrm{v})\\
&+2\left(a_2-\frac{33961}{181440}\,\mathrm{v}^2\right)\cos(2\,\mathrm{v})+2\left(a_1+\frac{280997}{181440}\,\mathrm{v}^2\right)\cos(\mathrm{v})+\frac{17273}{72576}\,\mathrm{v}^2\Bigg]\\
&\mathrm{v}\left(32\,a_4+\frac{125}{6}\,\mathrm{v}^2+18\,a_3+8\,a_2+2\,a_1\right)^{-3}
\end{aligned}
\tag{31}
$$

We demand that the phase-lag and its first, second and third derivatives to be equal to zero, i.e. we demand the satisfaction of the relations (28), (29), (30) and (31). Based on the above we obtain the coefficients mentioned in the Appendix C.

The behavior of the coefficients is given in the following Fig. 3.

The local truncation error of the new proposed method is given by:

$$
\mathrm{LTE}=-\frac{58061\,h^{12}}{31933440}\left(y_n^{(12)}+20\,\omega^6\,y_n^{(6)}+45\,\omega^8\,y_n^{(4)}+36\,\omega^{10}\,y_n^{(2)}+10\,\omega^{12}\,y_n\right) \tag{32}
$$

Remark 2 We can see that the local truncation error for the methods developed in paragraphs 3.2 and 3.3 is the same.

4 Comparative error analysis

We will study the following methods:

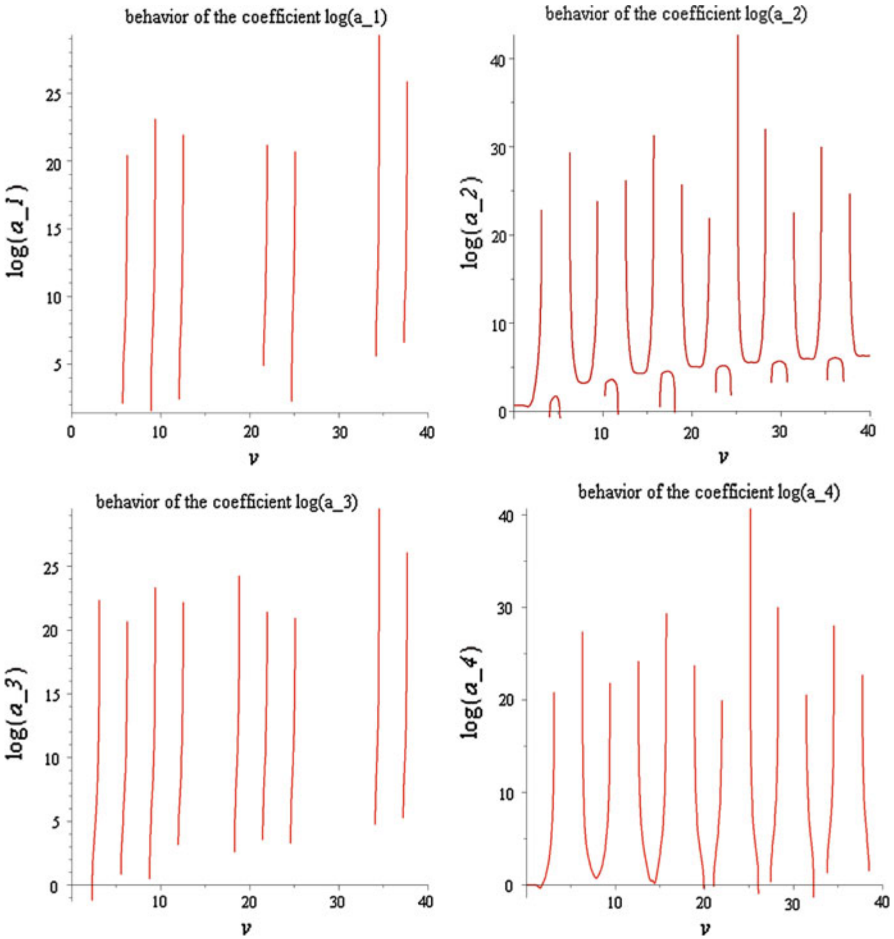

Fig. 3 Behavior of the coefficients of the new proposed method given by (66)–(69) for several values of v

– The eight-step ninth algebraic order method developed by Quinlan and Tremaine [154] which is indicated as **QT9**
– The ten-step eleventh algebraic order method developed by Quinlan and Tremaine [154] which is indicated as **QT11**
– The twelve-step thirteenth algebraic order method developed by Quinlan and Tremaine [154] which is indicated as **QT13**
– The classical eight-step method of the family of methods mentioned in paragraph 3 which is indicated as **CL**
– The method produced by Alolyan and Simos [155] which is indicated as **PLD1**
– The method produced by Alolyan and Simos [156] which is indicated as **PLD12**
– The new proposed method developed in paragraph 3.1 which is indicated as **PLD123a**

- The new proposed method developed in paragraph 3.2 which is indicated as **PLD123b**
- The new proposed method developed in paragraph 3.3 which is indicated as **PLD123c**

The radial time independent Schrödinger equation is of the form

$$y''(x) = f(x)\, y(x) \tag{33}$$

Based on the paper of Ixaru and Rizea [152], the function $f(x)$ can be written in the form:

$$f(x) = g(x) + G \tag{34}$$

where $g(x) = V(x) - V_c = g$, where V_c is the constant approximation of the potential and $G = v^2 = V_c - E$.

Now we express the derivatives $y_n^{(i)}$, $i = 2, 3, 4, \ldots$, which are terms of the local truncation error formulae, in terms of the Eq. 33. The expressions are presented as polynomials of G.

We substitute the expressions of the derivatives, produced in the previous step, into the local truncation error formulae.

Based on the procedure described above and on the formulae:

$$\begin{aligned}
y_n^{(2)} &= (\mathrm{V}(x) - V_c + G)\, \mathrm{y}(x) \\
y_n^{(4)} &= \left(\frac{d^2}{dx^2}\mathrm{V}(x)\right)\mathrm{y}(x) + 2\left(\frac{d}{dx}\mathrm{V}(x)\right)\left(\frac{d}{dx}\mathrm{y}(x)\right) \\
&\quad + (\mathrm{V}(x) - V_c + G)\left(\frac{d^2}{dx^2}\mathrm{y}(x)\right) \\
y_n^{(6)} &= \left(\frac{d^4}{dx^4}\mathrm{V}(x)\right)\mathrm{y}(x) + 4\left(\frac{d^3}{dx^3}\mathrm{V}(x)\right)\left(\frac{d}{dx}\mathrm{y}(x)\right) \\
&\quad + 3\left(\frac{d^2}{dx^2}\mathrm{V}(x)\right)\left(\frac{d^2}{dx^2}\mathrm{y}(x)\right) \\
&\quad + 4\left(\frac{d}{dx}\mathrm{V}(x)\right)^2 \mathrm{y}(x) \\
&\quad + 6\,(\mathrm{V}(x) - V_c + G)\left(\frac{d}{dx}\mathrm{y}(x)\right)\left(\frac{d}{dx}\mathrm{V}(x)\right) \\
&\quad + 4\,(\mathrm{U}(x) - V_c + G)\,\mathrm{y}(x)\left(\frac{d^2}{dx^2}\mathrm{V}(x)\right) \\
&\quad + (\mathrm{V}(x) - V_c + G)^2\left(\frac{d^2}{dx^2}\mathrm{y}(x)\right)\ldots
\end{aligned}$$

we obtain the expressions mentioned in Appendix C for the above methods.

We consider two cases in terms of the value of E:

– The Energy is close to the potential, i.e. $G = V_c - E \approx 0$. So only the free terms of the polynomials in G are considered. Thus for these values of G, the methods are of comparable accuracy. This is because the free terms of the polynomials in G, are the same for the cases of the classical method and of the new developed methods.
– $G \gg 0$ or $G \ll 0$. Then $\mid G \mid$ is a large number.
So, we have the following asymptotic expansions of the equations mentioned in [156] and in (71)–(73).

The eight-step ninth algebraic order method developed by Quinlan and Tremaine [154] (see for details in Alolyan and Simos [156])

$$\mathrm{LTE}_{\mathrm{QT9}} = h^{10} \left(-\frac{45767}{725760} \, \mathrm{y}(x) \, G^5 + \cdots \right) \tag{35}$$

The ten-step eleventh algebraic order method developed by Quinlan and Tremaine [154] (see for details in Alolyan and Simos [156])

$$\mathrm{LTE}_{\mathrm{QT11}} = h^{12} \left(-\frac{52559}{912384} \, \mathrm{y}(x) \, G^6 + \cdots \right) \tag{36}$$

The twelve-step thirteenth algebraic order method developed by Quinlan and Tremaine [154] (see for details in Alolyan and Simos [156])

$$\mathrm{LTE}_{\mathrm{QT13}} = h^{14} \left(-\frac{16301796103}{290594304000} \, \mathrm{y}(x) \, G^7 + \cdots \right) \tag{37}$$

The Classical Method of the Family[2] (see for details in Alolyan and Simos [156])

$$\mathrm{LTE}_{\mathrm{CL}} = h^{12} \left(-\frac{58061}{31933440} \, \mathrm{y}(x) \, G^6 + \cdots \right) \tag{38}$$

The method produced by Alolyan and Simos [155] (see Alolyan and Simos[155] for more details)

$$\begin{aligned} \mathrm{LTE}_{\mathrm{PLD1}} = h^{12} \Bigg[\Bigg(& \frac{987037}{31933440} \left(\frac{d^2}{dx^2} \, \mathrm{g}(x) \right) \mathrm{y}(x) \\ & + \frac{58061}{15966720} \left(\frac{d}{dx} \, \mathrm{g}(x) \right) \left(\frac{d}{dx} \, \mathrm{y}(x) \right) \\ & + \frac{58061}{31933440} \, \mathrm{g}(x)^2 \, \mathrm{y}(x) \Bigg) G^4 + \cdots \Bigg] \end{aligned} \tag{39}$$

[2] Classical method of the family is the method of the family with constant coefficients which has the same algebraic order.

The method produced by Alolyan and Simos [156] (see Appendix D for more details)

$$\mathrm{LTE}_{\mathrm{PLD12}} = h^{12}\left[\frac{58061}{7983360}\,G^4\left(\frac{d^2}{dx^2}\,\mathrm{g}(x)\right)\,\mathrm{y}(x)+\cdots\right] \tag{40}$$

The new proposed method developed in paragraph 3.1 (see Appendix D for more details)

$$\begin{aligned}\mathrm{LTE}_{\mathrm{PLD123a}} = h^{12}\Bigg[G^3\Bigg[&\frac{58061}{2661120}\left(\frac{d}{dx}g(x)\right)^2 y(x)\\&+\frac{58061}{3991680}\left(\frac{d^3}{dx^3}g(x)\right)\frac{d}{dx}y(x)+\frac{58061}{725760}\left(\frac{d^4}{dx^4}g(x)\right)y(x)\\&+\frac{58061}{1995840}\,g(x)\,y(x)\,\frac{d^2}{dx^2}g(x)\Bigg]+\cdots\Bigg]\end{aligned} \tag{41}$$

The new proposed methods developed in paragraphs 3.2 and 3.3 (see Appendix D for more details)

$$\begin{aligned}\mathrm{LTE}_{\mathrm{PLD123b}} = h^{12}\Bigg[G^3\Bigg[&\frac{58061}{177408}\left(\frac{d}{dx}g(x)\right)^2 y(x)\\&+\frac{58061}{133056}\,g(x)\,y(x)\,\frac{d^2}{dx^2}g(x)+\frac{1335403}{2661120}\left(\frac{d^4}{dx^4}g(x)\right)y(x)\\&+\frac{58061}{266112}\left(\frac{d^3}{dx^3}g(x)\right)\frac{d}{dx}y(x)\Bigg]+\cdots\Bigg]\end{aligned} \tag{42}$$

From the above equations we have the following theorem:

Theorem 2 – *For the eight-step ninth algebraic order method developed by Quinlan and Tremaine* [154] *the error increases as the fifth power of G*
- *For the ten-step eleventh algebraic order method developed by Quinlan and Tremaine* [154] *the error increases as the sixth power of G*
- *For the twelve-step thirteenth algebraic order method developed by Quinlan and Tremaine* [154] *the error increases as the seventh power of G*
- *For the Classical Method of the Family (see* [155] *for more details) the error increases as the sixth power of G*
- *For the method produced by Alolyan and Simos* [155] *the error increases as the fourth power of G*
- *For the method produced by Alolyan and Simos* [156] *the error increases as the fourth power of G but with smaller coefficient than the method of Alolyan and Simos* [155]
- *For the method developed in the paragraph 3.1 of this paper the error increases as the third power of G*
- *For the methods developed in the paragraphs 3.2 and 3.3 of this paper the error increases as the third power of G*

So, for the numerical solution of the time independent radial Schrödinger equation the new methods produced in this paper have the smallest error, especially for large values of $| G |=| V_c - E |$*, since they are of tenth algebraic order for which also the error increases as the third power of* G.

5 Stability analysis

The methods (8) with the coefficients (56)–(59), (61)–(64) and (66)– (69) are applied to the scalar test equation:

$$\psi'' = -t^2 \psi, \tag{43}$$

where $t \neq \omega$.

We thus obtain the following difference equation:

$$A_k(s)\,\psi_{n+k} + \cdots + A_1(s)\,\psi_{n+1} + A_0(s)\,\psi_n + A_1(s)\,\psi_{n-1} + \cdots + A_k(s)\,\psi_{n-k} = 0 \tag{44}$$

where $s = t\,h$, h is the step length and $A_0(s), A_1(s), \ldots, A_k(s)$ are polynomials of s.

The characteristic equation associated with (44) is given by:

$$A_k(s)\,\vartheta^k + \cdots + A_1(s)\,\vartheta + A_0(s) + A_1(s)\,\vartheta^{-1} + \cdots + A_k(s)\,\vartheta^{-k} = 0 \tag{45}$$

Definition 1 (see [46]) A symmetric $2k$-step method with the characteristic equation given by (45) is said to have an *interval of periodicity* $\left(0, s_0^2\right)$ if, for all $s \in \left(0, s_0^2\right)$, the roots z_i, $i = 1, 2$ satisfy

$$z_{1,2} = e^{\pm i\,\zeta(t\,h)}, \quad |z_i| \leq 1, \quad i = 3, 4 \tag{46}$$

where $\zeta(t\,h)$ is a real function of $t\,h$ and $s = t\,h$.

Definition 2 (see [46]) A method is called P-stable if its interval of periodicity is equal to $(0, \infty)$.

Definition 3 A method is called singularly almost P-stable if its interval of periodicity is equal to $(0, \infty) - S$[3] only when the frequency of the phase fitting is the same as the frequency of the scalar test equation, i.e. $mathrmv = s$.

In Fig. 4 we present the $s - \mathrm{v}$ plane for the methods developed in this paper. A shadowed area denotes the $s - \mathrm{v}$ region where the method is stable, while a white area denotes the region where the method is unstable.

In the case that the frequency of the scalar test equation is equal with the frequency of phase fitting, i.e. in the case that $\mathrm{v} = s$, and based on Fig. 4, it is easy to see that the interval of periodicity of the new methods is given by the following Table 1

[3] where S is a set of distinct points.

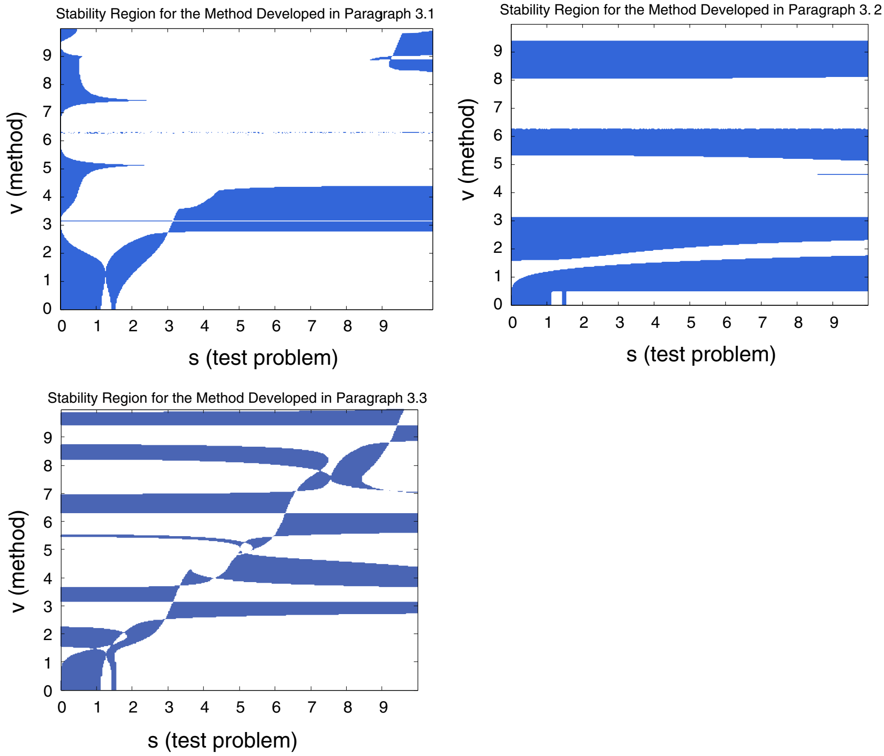

Fig. 4 $s - \text{v}$ plane of the New Methods produced in Sect. 3

Remark 3 For the solution of the Schrödinger equation the frequency of the exponential fitting is equal to the frequency of the scalar test equation. So, it is necessary to observe the surroundings of the first diagonal of the $s - v$ plane.

From the above analysis we have the following theorem:

Theorem 3 – *The method* (8) *with the coefficients* (56)–(59) *is of tenth algebraic order, has the phase-lag and its first, second and third derivatives equals to zero and has an interval of periodicity equals to:* $(0, 8.6)$.

- *The method* (8) *with the coefficients* (61)–(64) *is of tenth algebraic order, has the phase-lag and its first, second and third derivatives equals to zero and has an interval of periodicity equals to:* $(0, 3.3)$.
- *The method* (8) *with the coefficients* (66)–(69) *is of tenth algebraic order, has the phase-lag and its first, second and third derivatives equals to zero and has an interval of periodicity equals to:* $(0, 2.4)$.

Finally in Table 2 we present the characteristics of the methods developed and studied in this paper.

Table 1 Intervals of periodicity for the new developed methods

Method	Interval of periodicity
PLD123a	(0, 8.6)
PLD123b	(0, 3.3)
PLD123c	(0, 2.4)

Table 2 Basic characteristics of the methods developed and studied in this paper

Method	AO	Order of G	CFAE	IP
QT9	8	5	$-\frac{45767}{725760}$	(0, 0.52)
QT11	10	6	$-\frac{52559}{912384}$	(0, 0.17)
QT13	12	7	$-\frac{16301796103}{290594304000}$	(0, 0.046)
CL	10	6	$-\frac{58061}{31933440}$	(0, 1.3)
PLD1	10	4	$\frac{987037}{31933440}$	(0, 8.5264)
PLD12	10	4	$\frac{58061}{7983360}$	(0, 4.1)
PLD123a	10	3	$\frac{58061}{7983360}$	(0, 8.6)
PLD123b	10	3	$\frac{58061}{2661120}$	(0, 3.3)
PLD123c	10	3	$\frac{58061}{2661120}$	(0, 2.4)

We note that *AO* is the algebraic order, *CFAE* is the coefficient of the maximum power of G in the asymptotic expansion and order of G is the order of G in the asymptotic expansion of the local truncation error. *IP* is the interval of periodicity

6 Numerical results-conclusion

In order to illustrate the efficiency of the new methods obtained in Sect. 4, we apply them to the radial time independent Schrödinger equation.

In order to apply the new methods to the radial Schrödinger equation the value of parameter v is needed. For every problem of the one-dimensional Schrödinger equation given by (1) the parameter v is given by

$$v = \sqrt{|q(x)|} = \sqrt{|V(x) - E|} \tag{47}$$

where $V(x)$ is the potential and E is the energy.

6.1 Woods-Saxon potential

We use the well known Woods-Saxon potential given by

$$V(x) = \frac{u_0}{1+z} - \frac{u_0 z}{a\,(1+z)^2} \tag{48}$$

with $z = exp\,[(x - X_0)\,/a]$, $u_0 = -50$, $a = 0.6$, and $X_0 = 7.0$.

The behavior of Woods-Saxon potential is shown in the Fig. 5.

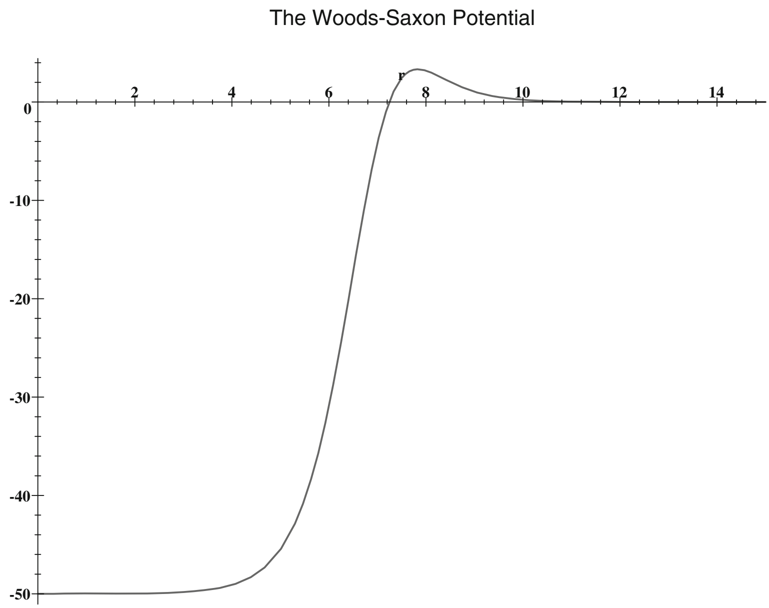

Fig. 5 The Woods-Saxon potential

It is well known that for some potentials, such as the Woods-Saxon potential, the definition of parameter v is not given as a function of x but it is based on some critical points which have been defined from the investigation of the appropriate potential (see for details [153]).

For the purpose of obtaining our numerical results it is appropriate to choose v as follows (see for details [153]):

$$v = \begin{cases} \sqrt{-50 + E}, & \text{for } x \in [0, 6.5 - 2h], \\ \sqrt{-37.5 + E}, & \text{for } x = 6.5 - h \\ \sqrt{-25 + E}, & \text{for } x = 6.5 \\ \sqrt{-12.5 + E}, & \text{for } x = 6.5 + h \\ \sqrt{E}, & \text{for } x \in [6.5 + 2h, 15] \end{cases} \tag{49}$$

6.2 Radial Schrödinger equation—the resonance problem

Consider the numerical solution of the radial time independent Schrödinger equation (1) in the well-known case of the Woods-Saxon potential (48). In order to solve this problem numerically we need to approximate the true (infinite) interval of integration by a finite interval. For the purpose of our numerical illustration we take the domain of integration as $x \in [0, 15]$. We consider equation (1) in a rather large domain of energies, i.e. $E \in [1, 1000]$.

In the case of positive energies, $E = k^2$, the potential dies away faster than the term $\frac{l(l+1)}{x^2}$ and the Schrödinger equation effectively reduces to

$$y''(x) + \left(k^2 - \frac{l(l+1)}{x^2}\right) y(x) = 0 \tag{50}$$

for x greater than some value X.

The above equation has linearly independent solutions $kxj_l(kx)$ and $kxn_l(kx)$ where $j_l(kx)$ and $n_l(kx)$ are the spherical Bessel and Neumann functions respectively. Thus the solution of equation (1) (when $x \rightarrow \infty$) has the asymptotic form

$$\begin{aligned} y(x) &\simeq Akxj_l(kx) - Bkxn_l(kx) \\ &\simeq AC\left[sin\left(kx - \frac{l\pi}{2}\right) + tan\delta_l cos\left(kx - \frac{l\pi}{2}\right)\right] \end{aligned} \tag{51}$$

where δ_l is the phase shift, that is calculated from the formula

$$tan\delta_l = \frac{y(x_2)S(x_1) - y(x_1)S(x_2)}{y(x_1)C(x_1) - y(x_2)C(x_2)} \tag{52}$$

for x_1 and x_2 distinct points in the asymptotic region (we choose x_1 as the right hand end point of the interval of integration and $x_2 = x_1 - h$) with $S(x) = kxj_l(kx)$ and $C(x) = -kxn_l(kx)$. Since the problem is treated as an initial-value problem, we need y_0, y_i, $i = 1(1)8$ before starting an eight-step method. From the initial condition we obtain y_0. The other values can be obtained using the Runge-Kutta-Nyström methods of Dormand et. al. (see [8]). With these starting values we evaluate at x_1 of the asymptotic region the phase shift δ_l.

For positive energies we have the so-called resonance problem. This problem consists either of finding the phase-shift δ_l or finding those E, for $E \in [1, 1000]$, at which $\delta_l = \frac{\pi}{2}$. We actually solve the latter problem, known as **the resonance problem** when the positive eigenenergies lie under the potential barrier.

The boundary conditions for this problem are:

$$y(0) = 0, \quad y(x) = \cos\left(\sqrt{E}x\right) \text{ for large} x. \tag{53}$$

We compute the approximate positive eigenenergies of the Woods-Saxon resonance problem using:

- The Numerov's method which is indicated as **Method I**
- The Exponentially-fitted two-step method developed by Raptis and Allison [47] which is indicated as **Method II**
- The Exponentially-fitted two-step P-stable method developed by Kalogitatou and Simos [49] which is indicated as **Method III**

– The Exponentially-fitted four-step method developed by Raptis [48] which is indicated as **Method IV**
– The eight-step ninth algebraic order method developed by Quinlan and Tremaine [154] which is indicated as **Method V**
– The ten-step eleventh algebraic order method developed by Quinlan and Tremaine [154] which is indicated as **Method VI**
– The twelve-step thirteenth algebraic order method developed by Quinlan and Tremaine [154] which is indicated as **Method VII**
– The classical eight-step method of the family of methods mentioned in Sect. 3 which is indicated as **Method VIII**
– The method produced by Alolyan and Simos [155] which is indicated as **Method IX**
– The method produced by Alolyan and Simos [156] which is indicated as **Method X**
– The method produced in paragraph 3.1 of this paper which is indicated as **Method XI**
– The method produced in paragraph 3.2 of this paper which is indicated as **Method XII**
– The method produced in paragraph 3.3 of this paper which is indicated as **Method XIII**

The computed eigenenergies are compared with exact ones. In Fig. 6 we present the maximum absolute error $log_{10}\,(Err)$ where

$$Err = |E_{calculated} - E_{accurate}| \tag{54}$$

of the eigenenergy $E_2 = 341.495874$, for several values of NFE = Number of Function Evaluations. In Fig. 7 we present the maximum absolute error $log_{10}\,(Err)$ where

$$Err = |E_{calculated} - E_{accurate}| \tag{55}$$

of the eigenenergy $E_3 = 989.701916$, for several values of NFE = Number of Function Evaluations.

7 Conclusions

In the present paper we have developed an eight-step method of tenth algebraic order with phase-lag and its first derivative equal to zero.

We have applied the new method to the resonance problem of the one-dimensional Schrödinger equation.

Based on the results presented above we have the following conclusions:

– The Exponentially-fitted two-step method developed by Raptis and Allison [47] (denoted as Method II) is more efficient than the Numerov's Method (denoted Method I) and for low number of function evaluations is more efficient than the Exponentially-fitted two-step P-stable method developed by Kalogitatou and Simos [49] (denoted as Method III).

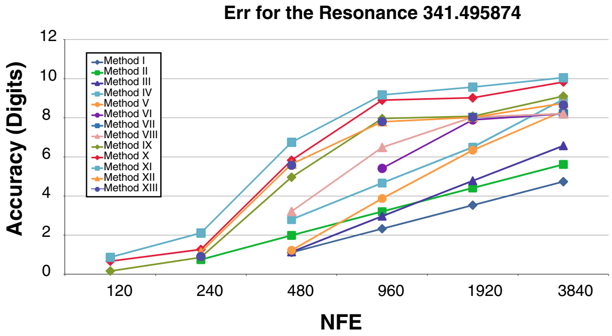

Fig. 6 Accuracy (Digits) for several values of NFE for the eigenvalue $E_2 = 341.495874$. The non-existence of a value of Accuracy (Digits) indicates that for this value of NFE, Accuracy (Digits) is less than 0

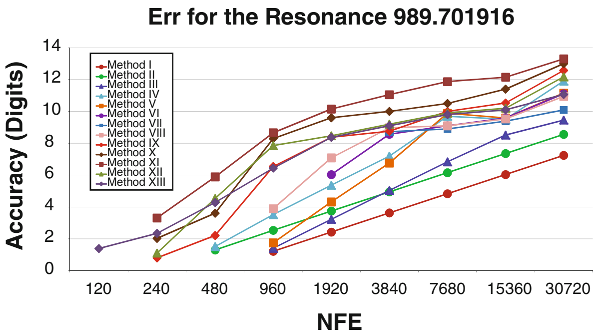

Fig. 7 Accuracy (Digits) for several values of NFE for the eigenvalue $E_3 = 989.701916$. The non-existence of a value of Accuracy (Digits) indicates that for this value of NFE, Accuracy (Digits) is less than 0

- The Exponentially-fitted two-step P-stable method developed by Kalogitatou and Simos [49] (denoted as Method III) is more efficient than the Exponentially-fitted two-step method developed by Raptis and Allison [47] (denoted as Method II) for high number of function evaluations.
- The Exponentially-fitted four-step method developed by Raptis [48] (denoted as Method IV) is more efficient than the Numerov' Method (denoted Method I), the Exponentially-fitted two-step method developed by Raptis and Allison [47] (denoted as Method II) and the Exponentially-fitted two-step P-stable method developed by Kalogitatou and Simos [49] (denoted as Method III)
- The eight-step ninth algebraic order method developed by Quinlan and Tremaine [154] (denoted as Method V) is more efficient than the Numerov' Method (denoted Method I), the Exponentially-fitted two-step method developed by Raptis and Allison [47] (denoted as Method II) and the Exponentially-fitted two-step P-stable method developed by Kalogitatou and Simos [49] (denoted as Method III) and less efficient than the Exponentially-fitted four-step method developed by Raptis [48] (denoted as Method IV)

- The ten-step eleventh algebraic order method developed by Quinlan and Tremaine [154] (denoted as Method VI) is more efficient than the Numerov' Method (denoted Method I), the Exponentially-fitted two-step method developed by Raptis and Allison [47] (denoted as Method II) and the Exponentially-fitted two-step P-stable method developed by Kalogitatou and Simos [49] (denoted as Method III) and the Exponentially-fitted four-step method developed by Raptis [48] (denoted as Method IV) for small number of function evaluations
- The twelve-step thirteenth algebraic order method developed by Quinlan and Tremaine [154] (denoted as Method VII) is more efficient than the Numerov' Method (denoted Method I), the Exponentially-fitted two-step method developed by Raptis and Allison [47] (denoted as Method II) and the Exponentially-fitted two-step P-stable method developed by Kalogitatou and Simos [49] (denoted as Method III), the Exponentially-fitted four-step method developed by Raptis [48] (denoted as Method IV) for small number of function evaluations, the eight-step ninth algebraic order method developed by Quinlan and Tremaine [154] (denoted as Method V) for small number of function evaluations and the ten-step eleventh algebraic order method developed by Quinlan and Tremaine [154] (denoted as Method VI) for small number of function evaluations
- The classical eight-step method of the family of methods mentioned in paragraph 4 (denoted as Method VIII) is more efficient than the Numerov' Method (denoted Method I), the Exponentially-fitted two-step method developed by Raptis and Allison [47] (denoted as Method II) and the Exponentially-fitted two-step P-stable method developed by Kalogitatou and Simos [49] (denoted as Method III), the Exponentially-fitted four-step method developed by Raptis [48] (denoted as Method IV) for small number of function evaluations, the eight-step ninth algebraic order method developed by Quinlan and Tremaine [154] (denoted as Method V) for small number of function evaluations, the ten-step eleventh algebraic order method developed by Quinlan and Tremaine [154] (denoted as Method VI) for small number of function evaluations and the twelve-step thirteenth algebraic order method developed by Quinlan and Tremaine [154] (denoted as Method VII) for small number of function evaluations
- The method produced by Alolyan and Simos [155] (denoted as Method IX) is more efficient than all the above mentioned methods
- The method produced by Alolyan and Simos [156] (denoted as Method X) is more efficient than all the above mentioned methods including the method developed in [155]
- The methods developed in paragraphs 3.2 and 3.3 (denoted as Methods XII and XIII respectively) have very good behavior for the low number of function evaluations, but they have very slow convergence and for this reason they more badly behaves than the Methods X and XI for high number of function evaluations.
- The methods developed in paragraph 3.1 (denoted as Method XI) is the most efficient one.

7.1 Summaries about the form of the numerical methods used for the solution of the radial Schrödinger equation

From the analysis presented above (comparative error analysis and comparative stability anslysis) and from the numerical results presented above, the following summary about the form of the numerical methods used for the approximate solution of the radial Schrödinger Equation is excluded:

Remark 4 When we try to develop a numerical method with vanishing phase-lag and its derivatives it is very important the construction of the method to be based on $b_i,\ i = 0, 1, \ldots$ i.e. on the coefficients of the right hand side of the method (8) and no on $a_i,\ i = 0, 1, \ldots$ i.e. on the coefficients of the left hand side of the method (8)

All computations were carried out on a IBM PC-AT compatible 80486 using double precision arithmetic with 16 significant digits accuracy (IEEE standard).

Appendix A

$$\begin{aligned}
b_1 = -\frac{1}{12}\Big[&135\,\mathrm{v} - 652\sin(\mathrm{v})\,\mathrm{v}^2\cos(\mathrm{v})^6 \\
&+ 599\sin(\mathrm{v})\,\mathrm{v}^2\cos(\mathrm{v})^4 - 54\sin(\mathrm{v})\,\mathrm{v}^2\cos(\mathrm{v})^2 \\
&- 348\sin(\mathrm{v})\,\mathrm{v}^2\cos(\mathrm{v})^5 - 48\sin(\mathrm{v})\,\mathrm{v}^2\cos(\mathrm{v})^7 + 224\sin(\mathrm{v})\,\mathrm{v}^2\cos(\mathrm{v})^8 \\
&+ 719\sin(\mathrm{v})\,\mathrm{v}^2\cos(\mathrm{v})^3 + 64\sin(\mathrm{v})\,\mathrm{v}^2\cos(\mathrm{v})^9 \\
&- 405\sin(\mathrm{v})\,\mathrm{v}^2\cos(\mathrm{v}) - 36\sin(\mathrm{v}) \\
&- 12\,\mathrm{v}^3 + 72\sin(\mathrm{v})\cos(\mathrm{v}) - 972\cos(\mathrm{v})^2\,\mathrm{v} \\
&- 1350\cos(\mathrm{v})^3\,\mathrm{v} + 1269\cos(\mathrm{v})^4\,\mathrm{v} \\
&+ 3240\cos(\mathrm{v})^5\,\mathrm{v} + 36\cos(\mathrm{v})^2\,\mathrm{v}^3 + 390\cos(\mathrm{v})^3\,\mathrm{v}^5 \\
&- 36\cos(\mathrm{v})^3\,\mathrm{v}^3 + 72\cos(\mathrm{v})^6\,\mathrm{v} \\
&- 2880\cos(\mathrm{v})^7\,\mathrm{v} - 36\cos(\mathrm{v})^4\,\mathrm{v}^3 + 360\cos(\mathrm{v})^5\,\mathrm{v}^5 \\
&+ 36\cos(\mathrm{v})^5\,\mathrm{v}^3 - 792\cos(\mathrm{v})^8\,\mathrm{v} \\
&+ 864\cos(\mathrm{v})^9\,\mathrm{v} + 12\cos(\mathrm{v})^6\,\mathrm{v}^3 + 450\cos(\mathrm{v})^4\,\mathrm{v}^5 \\
&- 12\cos(\mathrm{v})^7\,\mathrm{v}^3 + 288\cos(\mathrm{v})^{10}\,\mathrm{v} \\
&+ 120\cos(\mathrm{v})^6\,\mathrm{v}^5 - 99\sin(\mathrm{v})\,\mathrm{v}^2 - 432\sin(\mathrm{v})\cos(\mathrm{v})^5 + 288\sin(\mathrm{v})\cos(\mathrm{v})^2 \\
&- 120\sin(\mathrm{v})\cos(\mathrm{v})^3 + 1680\sin(\mathrm{v})\cos(\mathrm{v})^6 + 864\sin(\mathrm{v})\cos(\mathrm{v})^7 \\
&- 1164\sin(\mathrm{v})\cos(\mathrm{v})^4 - 768\sin(\mathrm{v})\cos(\mathrm{v})^8 \\
&- 384\sin(\mathrm{v})\cos(\mathrm{v})^9 + 270\cos(\mathrm{v})^2\,\mathrm{v}^5 \\
&+ 126\,\mathrm{v}\cos(\mathrm{v}) + 90\,\mathrm{v}^5\cos(\mathrm{v}) + 12\cos(\mathrm{v})\,\mathrm{v}^3\Big]\Big/ \\
&\Big[\mathrm{v}^5(\cos(\mathrm{v})^7 - \cos(\mathrm{v})^6 - 3\cos(\mathrm{v})^5 + 3\cos(\mathrm{v})^4 \\
&+ 3\cos(\mathrm{v})^3 - 3\cos(\mathrm{v})^2 - \cos(\mathrm{v}) + 1)\Big] \qquad (56)
\end{aligned}$$

$$b_2 = \frac{1}{12}\Big[90\,\mathrm{v} - 472\sin(\mathrm{v})\,\mathrm{v}^2\cos(\mathrm{v})^6$$
$$+ 486\sin(\mathrm{v})\,\mathrm{v}^2\cos(\mathrm{v})^4 - 78\sin(\mathrm{v})\,\mathrm{v}^2\cos(\mathrm{v})^2$$
$$- 452\sin(\mathrm{v})\,\mathrm{v}^2\cos(\mathrm{v})^5 + 144\sin(\mathrm{v})\,\mathrm{v}^2\cos(\mathrm{v})^7 + 144\sin(\mathrm{v})\,\mathrm{v}^2\cos(\mathrm{v})^8$$
$$+ 532\sin(\mathrm{v})\,\mathrm{v}^2\cos(\mathrm{v})^3 - 242\sin(\mathrm{v})\,\mathrm{v}^2\cos(\mathrm{v}) - 24\sin(\mathrm{v}) - 24\,\mathrm{v}^3 + 15\,\mathrm{v}^5$$
$$+ 48\sin(\mathrm{v})\cos(\mathrm{v}) - 684\cos(\mathrm{v})^2\,\mathrm{v} - 810\cos(\mathrm{v})^3\,\mathrm{v} + 1188\cos(\mathrm{v})^4\,\mathrm{v}$$
$$+ 2034\cos(\mathrm{v})^5\,\mathrm{v} + 72\cos(\mathrm{v})^2\,\mathrm{v}^3 + 285\cos(\mathrm{v})^3\,\mathrm{v}^5$$
$$- 72\cos(\mathrm{v})^3\,\mathrm{v}^3 - 738\cos(\mathrm{v})^6\,\mathrm{v}$$
$$- 1872\cos(\mathrm{v})^7\,\mathrm{v} - 72\cos(\mathrm{v})^4\,\mathrm{v}^3 + 90\cos(\mathrm{v})^5\,\mathrm{v}^5$$
$$+ 72\cos(\mathrm{v})^5\,\mathrm{v}^3 + 144\cos(\mathrm{v})^8\,\mathrm{v}$$
$$+ 576\cos(\mathrm{v})^9\,\mathrm{v} + 24\cos(\mathrm{v})^6\,\mathrm{v}^3 + 270\cos(\mathrm{v})^4\,\mathrm{v}^5$$
$$- 24\cos(\mathrm{v})^7\,\mathrm{v}^3 - 62\sin(\mathrm{v})\,\mathrm{v}^2$$
$$+ 120\sin(\mathrm{v})\cos(\mathrm{v})^5 + 216\sin(\mathrm{v})\cos(\mathrm{v})^2$$
$$- 168\sin(\mathrm{v})\cos(\mathrm{v})^3 + 1248\sin(\mathrm{v})\cos(\mathrm{v})^6$$
$$- 864\sin(\mathrm{v})\cos(\mathrm{v})^4 - 576\sin(\mathrm{v})\cos(\mathrm{v})^8 + 135\cos(\mathrm{v})^2\,\mathrm{v}^5 + 72\,\mathrm{v}\cos(\mathrm{v})$$
$$+ 45\,\mathrm{v}^5\cos(\mathrm{v}) + 24\cos(\mathrm{v})\,\mathrm{v}^3\Big]\Big/$$
$$\Big[\mathrm{v}^5(\cos(\mathrm{v})^7 - \cos(\mathrm{v})^6 - 3\cos(\mathrm{v})^5 + 3\cos(\mathrm{v})^4$$
$$+ 3\cos(\mathrm{v})^3 - 3\cos(\mathrm{v})^2 - \cos(\mathrm{v}) + 1)\Big] \tag{57}$$

$$b_3 = -\frac{1}{12}\Big[45\,\mathrm{v} - 56\sin(\mathrm{v})\,\mathrm{v}^2\cos(\mathrm{v})^6$$
$$+ 141\sin(\mathrm{v})\,\mathrm{v}^2\cos(\mathrm{v})^4 - 78\sin(\mathrm{v})\,\mathrm{v}^2\cos(\mathrm{v})^2$$
$$- 424\sin(\mathrm{v})\,\mathrm{v}^2\cos(\mathrm{v})^5 + 144\sin(\mathrm{v})\,\mathrm{v}^2\cos(\mathrm{v})^7 + 401\sin(\mathrm{v})\,\mathrm{v}^2\cos(\mathrm{v})^3$$
$$- 103\sin(\mathrm{v})\,\mathrm{v}^2\cos(\mathrm{v}) - 12\sin(\mathrm{v}) - 24\,\mathrm{v}^3$$
$$+ 24\sin(\mathrm{v})\cos(\mathrm{v}) - 360\cos(\mathrm{v})^2\,\mathrm{v}$$
$$- 306\cos(\mathrm{v})^3\,\mathrm{v} + 999\cos(\mathrm{v})^4\,\mathrm{v} + 504\cos(\mathrm{v})^5\,\mathrm{v}$$
$$+ 72\cos(\mathrm{v})^2\,\mathrm{v}^3 + 90\cos(\mathrm{v})^3\,\mathrm{v}^5$$
$$- 72\cos(\mathrm{v})^3\,\mathrm{v}^3 - 1044\cos(\mathrm{v})^6\,\mathrm{v} - 216\cos(\mathrm{v})^7\,\mathrm{v}$$
$$- 72\cos(\mathrm{v})^4\,\mathrm{v}^3 + 72\cos(\mathrm{v})^5\,\mathrm{v}^3$$
$$+ 360\cos(\mathrm{v})^8\,\mathrm{v} + 24\cos(\mathrm{v})^6\,\mathrm{v}^3 + 30\cos(\mathrm{v})^4\,\mathrm{v}^5$$
$$- 24\cos(\mathrm{v})^7\,\mathrm{v}^3 - 25\sin(\mathrm{v})\,\mathrm{v}^2$$
$$+ 528\sin(\mathrm{v})\cos(\mathrm{v})^5 + 144\sin(\mathrm{v})\cos(\mathrm{v})^2$$
$$- 264\sin(\mathrm{v})\cos(\mathrm{v})^3 + 192\sin(\mathrm{v})\cos(\mathrm{v})^6$$
$$- 288\sin(\mathrm{v})\cos(\mathrm{v})^7 - 324\sin(\mathrm{v})\cos(\mathrm{v})^4 + 90\cos(\mathrm{v})^2\,\mathrm{v}^5 + 18\,\mathrm{v}\cos(\mathrm{v})$$

$$+ 30\,v^5\cos(v) + 24\cos(v)\,v^3\Big] \Big/ \Big[v^5(\cos(v)^7 - \cos(v)^6 - 3\cos(v)^5 + 3\cos(v)^4 + 3\cos(v)^3 - 3\cos(v)^2 - \cos(v) + 1)\Big] \quad (58)$$

$$b_4 = \frac{1}{48}\Big[72\,v + 176\sin(v)\,v^2\cos(v)^6 - 408\sin(v)\,v^2\cos(v)^4 + 252\sin(v)\,v^2\cos(v)^2 - 176\sin(v)\,v^2\cos(v)^5 + 412\sin(v)\,v^2\cos(v)^3 - 254\sin(v)\,v^2\cos(v) - 24\sin(v) - 48\,v^3 + 15\,v^5 + 120\sin(v)\cos(v) - 486\cos(v)^2\,v + 522\cos(v)^3\,v + 702\cos(v)^4\,v - 720\cos(v)^5\,v + 144\cos(v)^2\,v^3 + 15\cos(v)^3\,v^5 - 144\cos(v)^3\,v^3 - 288\cos(v)^6\,v + 288\cos(v)^7\,v - 144\cos(v)^4\,v^3 + 144\cos(v)^5\,v^3 + 48\cos(v)^6\,v^3 - 48\cos(v)^7\,v^3 - 2\sin(v)\,v^2 + 192\sin(v)\cos(v)^5 - 72\sin(v)\cos(v)^2 - 312\sin(v)\cos(v)^3 - 192\sin(v)\cos(v)^6 + 288\sin(v)\cos(v)^4 + 45\cos(v)^2\,v^5 - 90\,v\cos(v) + 45\,v^5\cos(v) + 48\cos(v)\,v^3\Big] \Big/ \Big[v^5(\cos(v)^7 - \cos(v)^6 - 3\cos(v)^5 + 3\cos(v)^4 + 3\cos(v)^3 - 3\cos(v)^2 - \cos(v) + 1)\Big] \quad (59)$$

For small values of $|v|$ the formulae given by (56)–(59) are subject to heavy cancellations. In this case the following Taylor series expansions should be used:

$$b_1 = \frac{280997}{181440} - \frac{58061}{142560}\,v^2 + \frac{28712509}{266872320}\,v^4 - \frac{123514597}{7544275200}\,v^6 + \frac{54472029641}{35568742809600}\,v^8 - \frac{1759347854969}{15205637551104000}\,v^{10} + \frac{1158375152710183}{281000181944401920000}\,v^{12} - \frac{538227640391369}{138492946815455232000 0}\,v^{14} - \frac{441593667995985667}{2016457305633028177920000 0}\,v^{16} + \cdots$$

$$b_2 = -\frac{33961}{181440} + \frac{58061}{285120}\,v^2$$

$$
\begin{aligned}
& - \frac{13984933}{373621248} v^4 + \frac{294884453}{98075577600} v^6 \\
& - \frac{86102209}{725892710400} v^8 + \frac{21720638743}{2764661372928000} v^{10} \\
& + \frac{184186455588151}{281000181944401920000} v^{12} + \frac{8006349930637}{62951339461570560000} v^{14} \\
& + \frac{401632476631602791}{2016457305633028177920000} v^{16} + \cdots \\
b_3 = & \frac{173531}{181440} - \frac{58061}{997920} v^2 \\
& + \frac{7590455}{2615348736} v^4 + \frac{21770477}{98075577600} v^6 \\
& + \frac{1630632631}{35568742809600} v^8 + \frac{823818515167}{106439462857728000} v^{10} \\
& + \frac{347455118432089}{281000181944401920000} v^{12} + \frac{1827804753075907}{969450627708186624 0000} v^{14} \\
& + \frac{566908451171062 57}{2045681324555245977600000} v^{16} + \cdots \\
b_4 = & \frac{45767}{725760} + \frac{58061}{7983360} v^2 \\
& + \frac{52351801}{52306974720} v^4 + \frac{28777207}{196151155200} v^6 \\
& + \frac{281794861}{12934088294400} v^8 + \frac{2731484688089}{851515702861824000} v^{10} \\
& + \frac{522023359687903}{1124000727777607680000} v^{12} + \frac{2558127234111307}{38778025108327464960000} v^{14} \\
& + \frac{5190698721636537281}{56460804557724788981 7600000} v^{16} + \ldots
\end{aligned} \tag{60}
$$

Appendix B

$$
a_0 = \frac{1}{272160} \frac{T_9}{(\sin(v))^5}
$$

$$
\begin{aligned}
T_9 = & 747402 \sin(v) (\cos(v))^2 v^2 + 274602 \sin(v) (\cos(v))^6 v^2 \\
& - 1334808 \sin(v) (\cos(v))^4 v^2 + 7970094 \sin(v) (\cos(v))^3 \\
& - 718788 \sin(v) \cos(v) - 18227565 \sin(v) (\cos(v))^4 \\
& + 8802822 \sin(v) (\cos(v))^2 + 38202 \sin(v) v^2 \\
& + 274602 \sin(v) (\cos(v))^8 v^2 + 1532469 \sin(v) \\
& + 3805356 \sin(v) (\cos(v))^8 - 2082372 \sin(v) (\cos(v))^7 \\
& + 4079958 \sin(v) (\cos(v))^6 - 1759974 \sin(v) (\cos(v))^5
\end{aligned}
$$

$$
\begin{aligned}
&-718788\,\mathrm{v}+4055338\,(\cos(\mathrm{v}))^5\,\mathrm{v}\\
&-388291\,(\cos(\mathrm{v}))^7\,\mathrm{v}-2729617\,\mathrm{v}\,(\cos(\mathrm{v}))^3\\
&-9978474\,\mathrm{v}\,(\cos(\mathrm{v}))^2+76404\,\mathrm{v}\cos(\mathrm{v})\\
&+10973854\,\mathrm{v}\,(\cos(\mathrm{v}))^4-1006874\,(\cos(\mathrm{v}))^9\,\mathrm{v}\\
&-2297304\,(\cos(\mathrm{v}))^6\,\mathrm{v}-1388248\,(\cos(\mathrm{v}))^8\,\mathrm{v}
\end{aligned}
\tag{61}
$$

$$
\begin{aligned}
a_1 &= -\frac{1}{362880}\,\frac{T_{10}}{(\sin(\mathrm{v}))^5}\\
T_{10} &= 2644366\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^2+6676980\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^4\\
&+561994\,\sin(\mathrm{v})\,\mathrm{v}^2-5552992\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^6\\
&-5623400\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^3+8063267\,\sin(\mathrm{v})\cos(\mathrm{v})\\
&-12777511\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^5-359394\,\sin(\mathrm{v})\\
&+10330684\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^7+549204\,\cos(\mathrm{v})\,\mathrm{v}^2\sin(\mathrm{v})\\
&-366136\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^3\,\mathrm{v}^2-1123988\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^2\,\mathrm{v}^2\\
&-915340\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^5\,\mathrm{v}^2+561994\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^4\,\mathrm{v}^2\\
&+732272\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^7\,\mathrm{v}^2-6726308\,\mathrm{v}\cos(\mathrm{v})\\
&-4164744\,(\cos(\mathrm{v}))^7\,\mathrm{v}+176895\,\mathrm{v}\\
&+672982\,\mathrm{v}\,(\cos(\mathrm{v}))^3-377155\,\mathrm{v}\,(\cos(\mathrm{v}))^4\\
&+6809110\,(\cos(\mathrm{v}))^5\,\mathrm{v}+4213478\,(\cos(\mathrm{v}))^6\,\mathrm{v}-2562952\,(\cos(\mathrm{v}))^8\,\mathrm{v}\\
&-1443306\,\mathrm{v}\,(\cos(\mathrm{v}))^2
\end{aligned}
\tag{62}
$$

$$
\begin{aligned}
a_2 &= \frac{1}{181440}\,\frac{T_{11}}{(\sin(\mathrm{v}))^5}\\
T_{11} &= 2638388\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^2-7320467\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^4\\
&-1735310\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^5+2656698\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^3\\
&-469476\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^4\,\mathrm{v}^2+692263\,\sin(\mathrm{v})-239596\,\sin(\mathrm{v})\cos(\mathrm{v})\\
&+274602\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^6\,\mathrm{v}^2+3988424\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^6\\
&+79728\,\sin(\mathrm{v})\,\mathrm{v}^2+115146\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^2\,\mathrm{v}^2\\
&-1222511\,\mathrm{v}\,(\cos(\mathrm{v}))^3-3326158\,\mathrm{v}\,(\cos(\mathrm{v}))^2\\
&-239596\,\mathrm{v}+159456\,\mathrm{v}\cos(\mathrm{v})\\
&-869573\,(\cos(\mathrm{v}))^7\,\mathrm{v}-2082372\,(\cos(\mathrm{v}))^6\,\mathrm{v}\\
&+1934020\,(\cos(\mathrm{v}))^5\,\mathrm{v}+4966334\,\mathrm{v}\,(\cos(\mathrm{v}))^4
\end{aligned}
\tag{63}
$$

$$
\begin{aligned}
a_3 &= -\frac{1}{1088640}\,\frac{T_{12}}{(\sin(\mathrm{v}))^5}\\
T_{12} &= -16629852\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^3-359394\,\sin(\mathrm{v})\\
&-2082372\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^2\,\mathrm{v}^2+8297217\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^5
\end{aligned}
$$

$$
\begin{aligned}
&- 1098408\ \sin(v)(\cos(v))^3 v^2 + 3123558\ \sin(v)(\cos(v))^2\\
&+ 1041186\ \sin(v)(\cos(v))^4 v^2 + 8331243\ \sin(v)\cos(v)\\
&+ 549204\ \sin(v)(\cos(v))^5 v^2 - 2082372\ \sin(v)(\cos(v))^4\\
&+ 549204\ \cos(v) v^2 \sin(v) + 1041186\ \sin(v) v^2\\
&- 3817682\ (\cos(v))^5 v - 2113246\, v(\cos(v))^2 - 5767924\, v\cos(v)\\
&+ 444871\, v + 2859709\, v(\cos(v))^4\\
&- 1189942\ (\cos(v))^6 v + 8903814\, v(\cos(v))^3 \qquad (64)
\end{aligned}
$$

For small values of |v| the formulae given by (61)–(64) are subject to heavy cancellations. In this case the following Taylor series expansions should be used:

$$
\begin{aligned}
a_0 &= \frac{290305}{399168} v^6 - \frac{169692883}{373621248} v^8\\
&+ \frac{8141065577}{56043187200} v^{10} - \frac{2639480293129}{80029671321600} v^{12}\\
&+ \frac{1393168001723677}{319318388573184000} v^{14}\\
&- \frac{1921561482649313}{4323079722221568000} v^{16} + \cdots\\
a_1 &= -1 - \frac{290305}{532224} v^6 + \frac{30299}{101088} v^8\\
&- \frac{5656403939}{74724249600} v^{10} + \frac{424865451287}{53353114214400} v^{12}\\
&- \frac{27745110079079}{38705259220992000} v^{14} + \frac{363753574916891}{18733345462960128000} v^{16} + \cdots\\
a_2 &= 2 + \frac{58061}{266112} v^6 - \frac{88175239}{1245404160} v^8\\
&+ \frac{11780879}{3396556800} v^{10} - \frac{134835016429}{266765571072000} v^{12}\\
&- \frac{409794505823}{12522289747968000} v^{14} - \frac{1151574617491241}{187333454629601280000} v^{16} + \cdots\\
a_3 &= -2 - \frac{58061}{1596672} v^6 - \frac{1715227}{934053120} v^8\\
&- \frac{90457351}{224172748800} v^{10} - \frac{3578844721}{61561285632000} v^{12}\\
&- \frac{10885085691401}{1277273554292736000} v^{14} - \frac{342384733372831}{281000181944401920000} v^{16} + \cdots \qquad (65)
\end{aligned}
$$

Appendix C

$$
\begin{aligned}
a_1 &= -\frac{1}{1088640} \frac{T_{15}}{(\sin(v))^5 \left[24\ (\cos(v))^2 + 8\ (\cos(v))^4 + 3\right]}\\
T_{15} &= -3234546\ \sin(v) + 68623152\ \sin(v)(\cos(v))^6
\end{aligned}
$$

$$
\begin{aligned}
&+30347676\ \sin(v)(\cos(v))^2 v^2 - 5181900\ \sin(v)(\cos(v))^5 v^2\\
&+13487856\ \sin(v)(\cos(v))^6 v^2 + 13487856\ \sin(v)(\cos(v))^8 v^2\\
&+3109140\ \cos(v)\, v^2 \sin(v) - 2072760\ \sin(v)(\cos(v))^3 v^2\\
&-62381334\ \sin(v)(\cos(v))^4 v^2 + 4145520\ \sin(v)(\cos(v))^7 v^2\\
&-14821479\ \sin(v)(\cos(v))^5 + 8759940\ \sin(v)(\cos(v))^7\\
&+9331152\ \sin(v)(\cos(v))^3 - 10228496\ (\cos(v))^{10} v\\
&+2929088\ (\cos(v))^{12} v - 11105984\ (\cos(v))^{11} v\\
&+4393632\ \sin(v)(\cos(v))^{11} - 6625416\ \sin(v)(\cos(v))^9\\
&-989109\ \sin(v)\cos(v)\\
&+5057946\, v^2 \sin(v) + 32424750\ \sin(v)(\cos(v))^2\\
&-94700364\ \sin(v)(\cos(v))^4 - 26975712\ \sin(v)(\cos(v))^8\\
&-26034948\, v\cos(v) + 60182646\ (\cos(v))^5 v\\
&-3920706\ (\cos(v))^2 v + 46731846\ (\cos(v))^3 v\\
&+23064813\ (\cos(v))^4 v + 77220432\ (\cos(v))^9 v\\
&-43833938\ (\cos(v))^6 v + 30348464\ (\cos(v))^8 v\\
&-123131272\ (\cos(v))^7 v + 1592055\, v
\end{aligned}
\tag{66}
$$

$$
a_2 = \frac{1}{544320}\,\frac{T_{16}}{(\sin(v))^5\left[24\ (\cos(v))^2 + 8\ (\cos(v))^4 + 3\right]}
$$

$$
\begin{aligned}
T_{16} = {}&2870274\ \sin(v)(\cos(v))^2 v^2\\
&+2369634\ \sin(v)(\cos(v))^6 v^2 + 815064\ \sin(v)(\cos(v))^8 v^2\\
&-6619716\ \sin(v)(\cos(v))^4 v^2 + 100491\ \sin(v)\\
&+9978474\ (\cos(v))^2 v - 16658976\ (\cos(v))^{10} v\\
&+3295224\ (\cos(v))^{11} v + 8329488\ \sin(v)(\cos(v))^9\\
&+4393632\ \sin(v)(\cos(v))^{10} + 718788\ \sin(v)\cos(v)\\
&+564744\, v^2 \sin(v) + 1597884\ \sin(v)(\cos(v))^2\\
&-7970094\ \sin(v)(\cos(v))^3 + 2336577\ \sin(v)(\cos(v))^4\\
&-14328198\ \sin(v)(\cos(v))^5 - 6714768\ \sin(v)(\cos(v))^6\\
&-10612704\ \sin(v)(\cos(v))^7 - 1665096\ \sin(v)(\cos(v))^8\\
&+1129488\, v\cos(v) - 17498212\ (\cos(v))^5 v + 9563005\ (\cos(v))^3 v\\
&-4948714\ (\cos(v))^4 v - 2846296\ (\cos(v))^9 v\\
&+4823628\ (\cos(v))^6 v + 29949520\ (\cos(v))^8 v\\
&+6308071\ (\cos(v))^7 v + 718788\, v
\end{aligned}
\tag{67}
$$

$$
a_3 = -\frac{1}{362880}\,\frac{T_{17}}{(\sin(v))^5\left[24\ (\cos(v))^2 + 8\ (\cos(v))^4 + 3\right]}
$$

$$
\begin{aligned}
T_{17} = {} & 2196816\ (\cos(v))^{10}\, v - 5311271\ \sin(v)\,(\cos(v))^{5} \\
& - 5576062\ (\cos(v))^{6}\, v + 5454336\ (\cos(v))^{7}\, v \\
& - 359394\ \sin(v) + 345460\ \cos(v)\, v^{2} \sin(v) \\
& + 345460\ \sin(v)\,(\cos(v))^{5}\, v^{2} + 1041186\, v^{2} \sin(v) \\
& + 2776496\ \sin(v)\,(\cos(v))^{6}\, v^{2} + 5552992\ \sin(v)\,(\cos(v))^{8} \\
& + 3071708\ \sin(v)\,(\cos(v))^{3} + 1057040\ \sin(v)\,(\cos(v))^{6} \\
& - 6495522\ (\cos(v))^{5}\, v + 15978950\ (\cos(v))^{3}\, v \\
& - 12841294\ \sin(v)\,(\cos(v))^{4}\, v^{2} + 1038234\ (\cos(v))^{2}\, v \\
& + 177240\ (\cos(v))^{8}\, v + 2776496\ \sin(v)\,(\cos(v))^{8}\, v^{2} \\
& + 1830680\ \sin(v)\,(\cos(v))^{9} + 6247116\ \sin(v)\,(\cos(v))^{2}\, v^{2} \\
& + 4081942\ \sin(v)\,(\cos(v))^{2} + 271688\ \sin(v)\,(\cos(v))^{7} \\
& - 2776496\ (\cos(v))^{9}\, v - 690920\ \sin(v)\,(\cos(v))^{3}\, v^{2} \\
& - 20559460\ \sin(v)\,(\cos(v))^{4} + 444871\, v - 1934388\, v \cos(v) \\
& + 158075\ \sin(v)\cos(v) + 1698021\ (\cos(v))^{4}\, v
\end{aligned} \tag{68}
$$

$$
\begin{aligned}
a_4 = {} & -\frac{1}{544320}\,\frac{T_{18}}{(\sin(v))^{5}\left[24\ (\cos(v))^{2} + 8\ (\cos(v))^{4} + 3\right]} \\
T_{18} = {} & -718788\ \sin(v)\cos(v) + 274602\ \sin(v)\,(\cos(v))^{8}\, v^{2} \\
& + 747402\ \sin(v)\,(\cos(v))^{2}\, v^{2} + 38202\, v^{2} \sin(v) \\
& - 994938\ \sin(v)\,(\cos(v))^{2} + 7970094\ \sin(v)\,(\cos(v))^{3} \\
& + 1912275\ \sin(v)\,(\cos(v))^{4} - 1759974\ \sin(v)\,(\cos(v))^{5} \\
& - 274602\ \sin(v)\,(\cos(v))^{6} - 2082372\ \sin(v)\,(\cos(v))^{7} \\
& - 549204\ \sin(v)\,(\cos(v))^{8} - 1334808\ \sin(v)\,(\cos(v))^{4}\, v^{2} \\
& + 274602\ \sin(v)\,(\cos(v))^{6}\, v^{2} - 100491\ \sin(v) \\
& + 76404\, v \cos(v) + 4055338\ (\cos(v))^{5}\, v \\
& - 9978474\ (\cos(v))^{2}\, v - 2729617\ (\cos(v))^{3}\, v + 10973854\ (\cos(v))^{4}\, v \\
& - 1006874\ (\cos(v))^{9}\, v - 718788\, v - 2297304\ (\cos(v))^{6}\, v \\
& - 1388248\ (\cos(v))^{8}\, v - 388291\ (\cos(v))^{7}\, v
\end{aligned} \tag{69}
$$

For small values of $|v|$ the formulae given by (66)–(69) are subject to heavy cancellations. In this case the following Taylor series expansions should be used:

$$
\begin{aligned}
a_1 = {} & -1 + \frac{58061}{1596672}\, v^{6} - \frac{72117781}{3269185920}\, v^{8} \\
& + \frac{479082233603}{54922323456000}\, v^{10} - \frac{40122209945659}{7842907789516800}\, v^{12}
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \frac{10092789713026871}{3681553185902592000}\,\mathrm{v}^{14} \\
&\quad - \frac{17875413152997844237}{1927661248138597171200 0}\,\mathrm{v}^{16} + \cdots \\
a_2 &= 2 - \frac{58061}{798336}\,\mathrm{v}^{6} + \frac{145183373}{5230697472}\,\mathrm{v}^{8} \\
&\quad - \frac{96718267577}{27461161728000}\,\mathrm{v}^{10} - \frac{13225008590839}{7842907789516800}\,\mathrm{v}^{12} \\
&\quad - \frac{265046587303349}{240716939078246400 0}\,\mathrm{v}^{14} \\
&\quad + \frac{34153996668417964 99}{3855322496277194342400 0}\,\mathrm{v}^{16} + \cdots \\
a_3 &= -2 + \frac{58061}{1241856}\,\mathrm{v}^{6} - \frac{315937}{1017080064}\,\mathrm{v}^{8} \\
&\quad - \frac{141118553533}{42717362688000}\,\mathrm{v}^{10} - \frac{398404268309}{225927384883200}\,\mathrm{v}^{12} \\
&\quad - \frac{40492887490178023}{48678314346934272000}\,\mathrm{v}^{14} \\
&\quad - \frac{135258641710069849}{384433867150147584000}\,\mathrm{v}^{16} + \cdots \\
a_4 &= 1 - \frac{58061}{5588352}\,\mathrm{v}^{6} - \frac{985953659}{183074411520}\,\mathrm{v}^{8} \\
&\quad - \frac{364726453673}{192228132096000}\,\mathrm{v}^{10} - \frac{28853084317627}{54900354526617600}\,\mathrm{v}^{12} \\
&\quad - \frac{19282473212874713}{219052414561204224000}\,\mathrm{v}^{14} \\
&\quad + \frac{518362675814354542 3}{269872574739403603968000}\,\mathrm{v}^{16} + \cdots
\end{aligned} \tag{70}
$$

Appendix D

The method produced by Alolyan and Simos [156]

$$
\begin{aligned}
\mathrm{LTE}_{\mathrm{PLD12}} = h^{12} \Bigg[& \frac{58061}{7983360}\, G^4 \left(\frac{d^2}{dx^2}\, \mathrm{g}(x) \right) \mathrm{y}(x) \\
& + G^3 \Bigg[\frac{987037}{6386688} \left(\frac{d^4}{dx^4} g\,(x) \right) y\,(x) + \frac{58061}{31933440}\, (g\,(x))^3\, y\,(x) \\
& + \frac{58061}{1140480} \left(\frac{d^3}{dx^3} g\,(x) \right) \frac{d}{dx} y\,(x) + \frac{58061}{5322240}\, g\,(x) \left(\frac{d}{dx} y\,(x) \right) \frac{d}{dx} g\,(x) \\
& + \frac{58061}{798336} \left(\frac{d}{dx} g\,(x) \right)^2 y\,(x) + \frac{58061}{580608}\, g\,(x)\, y\,(x)\, \frac{d^2}{dx^2} g\,(x) \Bigg]
\end{aligned}
$$

$$+G^2\left[\frac{9115577}{5322240}\left(\frac{d}{dx}g(x)\right)y(x)\frac{d^3}{dx^3}g(x)\right.$$

$$+\frac{58061}{83160}\left(\frac{d}{dx}g(x)\right)\left(\frac{d}{dx}y(x)\right)\frac{d^2}{dx^2}g(x)$$

$$+\frac{58061}{120960}g(x)y(x)\left(\frac{d}{dx}g(x)\right)^2+\frac{58061}{887040}(g(x))^2\left(\frac{d}{dx}y(x)\right)\frac{d}{dx}g(x)$$

$$+\frac{58061}{166320}g(x)\left(\frac{d}{dx}y(x)\right)\frac{d^3}{dx^3}g(x)+\frac{58061}{76032}g(x)y(x)\frac{d^4}{dx^4}g(x)$$

$$+\frac{1799891}{6386688}\left(\frac{d^6}{dx^6}g(x)\right)y(x)+\frac{4238453}{15966720}\left(\frac{d^5}{dx^5}g(x)\right)\frac{d}{dx}y(x)$$

$$+\frac{35591393}{31933440}\left(\frac{d^2}{dx^2}g(x)\right)^2y(x)+\frac{58061}{10644480}(g(x))^4y(x)$$

$$\left.+\frac{1799891}{5322240}(g(x))^2y(x)\frac{d^2}{dx^2}g(x)\right]+G$$

$$\left[\frac{58061}{22176}g(x)\left(\frac{d}{dx}y(x)\right)\left(\frac{d}{dx}g(x)\right)\frac{d^2}{dx^2}g(x)\right.$$

$$+\frac{26533877}{5322240}g(x)y(x)\left(\frac{d}{dx}g(x)\right)\frac{d^3}{dx^3}g(x)$$

$$+\frac{57886817}{15966720}\left(\frac{d^2}{dx^2}g(x)\right)y(x)\frac{d^4}{dx^4}g(x)$$

$$+\frac{754793}{199584}\left(\frac{d^2}{dx^2}g(x)\right)\left(\frac{d}{dx}y(x)\right)\frac{d^3}{dx^3}g(x)$$

$$+\frac{3773965}{1596672}\left(\frac{d}{dx}g(x)\right)\left(\frac{d}{dx}y(x)\right)\frac{d^4}{dx^4}g(x)$$

$$+\frac{290305}{145152}\left(\frac{d}{dx}g(x)\right)y(x)\frac{d^5}{dx^5}g(x)$$

$$+\frac{33036709}{7983360}\left(\frac{d}{dx}g(x)\right)^2y(x)\frac{d^2}{dx^2}g(x)$$

$$+\frac{22701851}{31933440}g(x)y(x)\frac{d^6}{dx^6}g(x)$$

$$+\frac{58061}{66528}(g(x))^2y(x)\left(\frac{d}{dx}g(x)\right)^2$$

$$+\frac{58061}{532224}(g(x))^3\left(\frac{d}{dx}y(x)\right)\frac{d}{dx}g(x)+\frac{2496623}{31933440}\left(\frac{d^8}{dx^8}g(x)\right)y(x)$$

$$+\frac{58061}{88704}\left(\frac{d}{dx}g(x)\right)^3\frac{d}{dx}y(x)$$

$$+\frac{754793}{3991680}\left(\frac{d^7}{dx^7}g(x)\right)\frac{d}{dx}y(x)$$
$$+\frac{8767211}{3991680}\left(\frac{d^3}{dx^3}g(x)\right)^2 y(x)+\frac{58061}{10644480}(g(x))^5 y(x)$$
$$+\frac{1335403}{3193344}(g(x))^3 y(x)\frac{d^2}{dx^2}g(x)+\frac{58061}{50688}(g(x))^2 y(x)\frac{d^4}{dx^4}g(x)$$
$$+\frac{102709909}{31933440}g(x)\,y(x)\left(\frac{d^2}{dx^2}g(x)\right)^2$$
$$+\frac{12831481}{15966720}g(x)\left(\frac{d}{dx}y(x)\right)\frac{d^5}{dx^5}g(x)$$
$$+\left.\frac{58061}{88704}(g(x))^2\left(\frac{d}{dx}y(x)\right)\frac{d^3}{dx^3}g(x)\right]$$
$$+\frac{20495533}{3193344}\left(\frac{d}{dx}g(x)\right)y(x)\left(\frac{d^3}{dx^3}g(x)\right)$$
$$\frac{d^2}{dx^2}g(x)+\frac{754793}{181440}g(x)\,y(x)\left(\frac{d^4}{dx^4}g(x)\right)\frac{d^2}{dx^2}g(x)$$
$$+\frac{58061}{31933440}\left(\frac{d^{10}}{dx^{10}}g(x)\right)y(x)+\frac{58061}{3193344}\left(\frac{d^9}{dx^9}g(x)\right)\frac{d}{dx}y(x)$$
$$+\frac{58061}{152064}\left(\frac{d^4}{dx^4}g(x)\right)^2 y(x)+\frac{290305}{236544}\left(\frac{d^2}{dx^2}g(x)\right)^3 y(x)$$
$$+\frac{58061}{249480}g(x)\left(\frac{d}{dx}y(x)\right)\frac{d^7}{dx^7}g(x)$$
$$+\frac{43139323}{7983360}g(x)\,y(x)\left(\frac{d}{dx}g(x)\right)^2\frac{d^2}{dx^2}g(x)$$
$$+\frac{58061}{114048}\left(\frac{d}{dx}g(x)\right)^4 y(x)$$
$$+\frac{58061}{31933440}(g(x))^6 y(x)+\frac{754793}{1596672}(g(x))^3 y(x)\left(\frac{d}{dx}g(x)\right)^2$$
$$+\frac{290305}{266112}g(x)\left(\frac{d}{dx}y(x)\right)\left(\frac{d}{dx}g(x)\right)^3$$
$$+\frac{58061}{1064448}(g(x))^4\left(\frac{d}{dx}y(x)\right)\frac{d}{dx}g(x)$$
$$+\frac{1103159}{591360}\left(\frac{d^2}{dx^2}g(x)\right)\left(\frac{d}{dx}y(x)\right)\frac{d^5}{dx^5}g(x)$$
$$+\frac{754793}{3193344}\left(\frac{d}{dx}g(x)\right)y(x)\frac{d^7}{dx^7}g(x)$$

$$+\frac{58061}{22176}\left(\frac{d^3}{dx^3}g(x)\right)\left(\frac{d}{dx}y(x)\right)\frac{d^4}{dx^4}g(x)$$
$$+\frac{1799891}{2661120}\left(\frac{d^3}{dx^3}g(x)\right)y(x)\frac{d^5}{dx^5}g(x)$$
$$+\frac{58061}{25344}\left(\frac{d}{dx}g(x)\right)^2 y(x)\frac{d^4}{dx^4}g(x)$$
$$+\frac{13876579}{31933440}(g(x))^2 y(x)\frac{d^6}{dx^6}g(x)$$
$$+\frac{1335403}{15966720}g(x)\,y(x)\frac{d^8}{dx^8}g(x)$$
$$+\frac{2148257}{3991680}(g(x))^3 y(x)\frac{d^4}{dx^4}g(x)$$
$$+\frac{2496623}{997920}g(x)\,y(x)\left(\frac{d^3}{dx^3}g(x)\right)^2$$
$$+\frac{9115577}{15966720}(g(x))^2\left(\frac{d}{dx}y(x)\right)\frac{d^5}{dx^5}g(x)$$
$$+\frac{69731261}{31933440}(g(x))^2 y(x)\left(\frac{d^2}{dx^2}g(x)\right)^2$$
$$+\frac{1103159}{6386688}(g(x))^4 y(x)\frac{d^2}{dx^2}g(x)$$
$$+\frac{290305}{798336}(g(x))^3\left(\frac{d}{dx}y(x)\right)\frac{d^3}{dx^3}g(x)$$
$$+\frac{406427}{456192}\left(\frac{d}{dx}g(x)\right)\left(\frac{d}{dx}y(x)\right)\frac{d^6}{dx^6}g(x)$$
$$+\frac{987037}{2128896}\left(\frac{d^2}{dx^2}g(x)\right)y(x)\frac{d^6}{dx^6}g(x)$$
$$+\frac{1799891}{354816}\left(\frac{d}{dx}g(x)\right)\left(\frac{d}{dx}y(x)\right)\left(\frac{d^2}{dx^2}g(x)\right)^2$$
$$+\frac{6328649}{1596672}\left(\frac{d}{dx}g(x)\right)^2\left(\frac{d}{dx}y(x)\right)\frac{d^3}{dx^3}g(x)$$
$$+\frac{290305}{133056}(g(x))^2\left(\frac{d}{dx}y(x)\right)\left(\frac{d}{dx}g(x)\right)$$
$$\frac{d^2}{dx^2}g(x)+\frac{18173093}{5322240}(g(x))^2 y(x)\left(\frac{d}{dx}g(x)\right)$$
$$\frac{d^3}{dx^3}g(x)+\frac{1335403}{399168}g(x)\left(\frac{d}{dx}y(x)\right)\left(\frac{d^4}{dx^4}g(x)\right)\frac{d}{dx}g(x)$$

$$+\frac{4238453}{798336}\,g(x)\left(\frac{d}{dx}y(x)\right)\left(\frac{d^3}{dx^3}g(x)\right)\frac{d^2}{dx^2}g(x)$$
$$+\frac{18753703}{7983360}\,g(x)\,y(x)\left(\frac{d^5}{dx^5}g(x)\right)\frac{d}{dx}g(x)\Bigg] \tag{71}$$

The new proposed method developed in paragraph 3.1

$$\mathrm{LTE}_{\mathrm{PLD123a}} = h^{12}\Bigg[G^3\Bigg[\frac{58061}{2661120}\left(\frac{d}{dx}g(x)\right)^2 y(x)$$
$$+\frac{58061}{3991680}\left(\frac{d^3}{dx^3}g(x)\right)\frac{d}{dx}y(x)$$
$$+\frac{58061}{725760}\left(\frac{d^4}{dx^4}g(x)\right)y(x)+\frac{58061}{1995840}\,g(x)\,y(x)\,\frac{d^2}{dx^2}g(x)\Bigg]$$
$$+G^2\Bigg[\frac{754793}{1995840}\left(\frac{d}{dx}g(x)\right)\left(\frac{d}{dx}y(x)\right)\frac{d^2}{dx^2}g(x)$$
$$+\frac{1799891}{1451520}\left(\frac{d}{dx}g(x)\right)y(x)\,\frac{d^3}{dx^3}g(x)$$
$$+\frac{2148257}{7983360}\,g(x)\,y(x)\left(\frac{d}{dx}g(x)\right)^2$$
$$+\frac{987037}{5322240}\,(g(x))^2\,y(x)\,\frac{d^2}{dx^2}g(x)$$
$$+\frac{2148257}{3991680}\,g(x)\,y(x)\,\frac{d^4}{dx^4}g(x)$$
$$+\frac{754793}{3991680}\,g(x)\left(\frac{d}{dx}y(x)\right)\frac{d^3}{dx^3}g(x)$$
$$+\frac{58061}{2661120}\,(g(x))^2\left(\frac{d}{dx}y(x)\right)\frac{d}{dx}g(x)+\frac{58061}{31933440}\,(g(x))^4\,y(x)$$
$$+\frac{987037}{5322240}\left(\frac{d^5}{dx^5}g(x)\right)\frac{d}{dx}y(x)+\frac{8651089}{10644480}\left(\frac{d^2}{dx^2}g(x)\right)^2 y(x)$$
$$+\frac{2496623}{10644480}\left(\frac{d^6}{dx^6}g(x)\right)y(x)\Bigg]+G\Bigg[\frac{34894661}{7983360}\,g(x)\,y(x)$$
$$\left(\frac{d^3}{dx^3}g(x)\right)\frac{d}{dx}g(x)+\frac{58061}{28512}\,g(x)\left(\frac{d}{dx}y(x)\right)\left(\frac{d^2}{dx^2}g(x)\right)\frac{d}{dx}g(x)$$
$$+\frac{4238453}{1140480}\left(\frac{d}{dx}g(x)\right)^2 y(x)\,\frac{d^2}{dx^2}g(x)$$
$$+\frac{58061}{28512}\left(\frac{d}{dx}g(x)\right)\left(\frac{d}{dx}y(x)\right)\frac{d^4}{dx^4}g(x)$$
$$+\frac{2148257}{1140480}\left(\frac{d}{dx}g(x)\right)y(x)\,\frac{d^5}{dx^5}g(x)$$

$$+\frac{1103159}{1596672}\,(g(x))^2\,y(x)\left(\frac{d}{dx}g(x)\right)^2$$
$$+\frac{987037}{997920}\,(g(x))^2\,y(x)\,\frac{d^4}{dx^4}g(x)$$
$$+\frac{58061}{177408}\,(g(x))^3\,y(x)\,\frac{d^2}{dx^2}g(x)$$
$$+\frac{4586819}{1330560}\left(\frac{d^2}{dx^2}g(x)\right)y(x)\,\frac{d^4}{dx^4}g(x)$$
$$+\frac{290305}{88704}\left(\frac{d^2}{dx^2}g(x)\right)\left(\frac{d}{dx}y(x)\right)\frac{d^3}{dx^3}g(x)$$
$$+\frac{1103159}{1596672}\,g(x)\left(\frac{d}{dx}y(x)\right)\frac{d^5}{dx^5}g(x)$$
$$+\frac{45229519}{15966720}\,g(x)\,y(x)\left(\frac{d^2}{dx^2}g(x)\right)^2$$
$$+\frac{58061}{114048}\,(g(x))^2\left(\frac{d}{dx}y(x)\right)\frac{d^3}{dx^3}g(x)$$
$$+\frac{58061}{798336}\,(g(x))^3\left(\frac{d}{dx}y(x)\right)\frac{d}{dx}g(x)$$
$$+\frac{10509041}{15966720}\,g(x)\,y(x)\,\frac{d^6}{dx^6}g(x)$$
$$+\frac{58061}{15966720}\,(g(x))^5\,y(x)+\frac{58061}{760320}\left(\frac{d^8}{dx^8}g(x)\right)y(x)$$
$$+\frac{58061}{114048}\left(\frac{d}{dx}g(x)\right)^3\frac{d}{dx}y(x)+\frac{58061}{27720}\left(\frac{d^3}{dx^3}g(x)\right)^2y(x)$$
$$+\frac{58061}{332640}\left(\frac{d^7}{dx^7}g(x)\right)\frac{d}{dx}y(x)\Bigg]$$
$$+\frac{1103159}{591360}\left(\frac{d^2}{dx^2}g(x)\right)\left(\frac{d}{dx}y(x)\right)\frac{d^5}{dx^5}g(x)$$
$$+\frac{406427}{456192}\left(\frac{d}{dx}g(x)\right)\left(\frac{d}{dx}y(x)\right)\frac{d^6}{dx^6}g(x)$$
$$+\frac{1799891}{354816}\left(\frac{d}{dx}g(x)\right)\left(\frac{d}{dx}y(x)\right)\left(\frac{d^2}{dx^2}g(x)\right)^2$$
$$+\frac{6328649}{1596672}\left(\frac{d}{dx}g(x)\right)^2\left(\frac{d}{dx}y(x)\right)\frac{d^3}{dx^3}g(x)$$
$$+\frac{1799891}{2661120}\left(\frac{d^3}{dx^3}g(x)\right)y(x)\,\frac{d^5}{dx^5}g(x)$$
$$+\frac{987037}{2128896}\left(\frac{d^2}{dx^2}g(x)\right)y(x)\,\frac{d^6}{dx^6}g(x)$$

$$+\frac{58061}{22176}\left(\frac{d^3}{dx^3}g(x)\right)\left(\frac{d}{dx}y(x)\right)\frac{d^4}{dx^4}g(x)$$
$$+\frac{58061}{114048}\left(\frac{d}{dx}g(x)\right)^4 y(x)$$
$$+\frac{58061}{31933440}(g(x))^6 y(x)+\frac{754793}{1596672}(g(x))^3 y(x)\left(\frac{d}{dx}g(x)\right)^2$$
$$+\frac{290305}{266112}g(x)\left(\frac{d}{dx}y(x)\right)\left(\frac{d}{dx}g(x)\right)^3$$
$$+\frac{58061}{1064448}(g(x))^4\left(\frac{d}{dx}y(x)\right)\frac{d}{dx}g(x)$$
$$+\frac{754793}{3193344}\left(\frac{d}{dx}g(x)\right)y(x)\frac{d^7}{dx^7}g(x)$$
$$+\frac{58061}{31933440}\left(\frac{d^{10}}{dx^{10}}g(x)\right)y(x)$$
$$+\frac{58061}{3193344}\left(\frac{d^9}{dx^9}g(x)\right)\frac{d}{dx}y(x)+\frac{290305}{236544}\left(\frac{d^2}{dx^2}g(x)\right)^3 y(x)$$
$$+\frac{58061}{152064}\left(\frac{d^4}{dx^4}g(x)\right)^2 y(x)+\frac{13876579}{31933440}(g(x))^2 y(x)\frac{d^6}{dx^6}g(x)$$
$$+\frac{754793}{181440}g(x)y(x)\left(\frac{d^4}{dx^4}g(x)\right)\frac{d^2}{dx^2}g(x)$$
$$+\frac{1335403}{399168}g(x)\left(\frac{d}{dx}y(x)\right)\left(\frac{d^4}{dx^4}g(x)\right)\frac{d}{dx}g(x)$$
$$+\frac{4238453}{798336}g(x)\left(\frac{d}{dx}y(x)\right)\left(\frac{d^3}{dx^3}g(x)\right)\frac{d^2}{dx^2}g(x)$$
$$+\frac{58061}{25344}\left(\frac{d}{dx}g(x)\right)^2 y(x)\frac{d^4}{dx^4}g(x)+\frac{1335403}{15966720}g(x)y(x)\frac{d^8}{dx^8}g(x)$$
$$+\frac{9115577}{15966720}(g(x))^2\left(\frac{d}{dx}y(x)\right)\frac{d^5}{dx^5}g(x)$$
$$+\frac{69731261}{31933440}(g(x))^2 y(x)\left(\frac{d^2}{dx^2}g(x)\right)^2$$
$$+\frac{58061}{249480}g(x)\left(\frac{d}{dx}y(x)\right)\frac{d^7}{dx^7}g(x)$$
$$+\frac{2496623}{997920}g(x)y(x)\left(\frac{d^3}{dx^3}g(x)\right)^2$$
$$+\frac{290305}{798336}(g(x))^3\left(\frac{d}{dx}y(x)\right)\frac{d^3}{dx^3}g(x)$$
$$+\frac{1103159}{6386688}(g(x))^4 y(x)\frac{d^2}{dx^2}g(x)$$

$$
\begin{aligned}
&+\frac{2148257}{3991680}\,(g(x))^3\,y(x)\,\frac{d^4}{dx^4}g(x)\\
&+\frac{18753703}{7983360}\,g(x)\,y(x)\left(\frac{d^5}{dx^5}g(x)\right)\frac{d}{dx}g(x)\\
&+\frac{290305}{133056}\,(g(x))^2\left(\frac{d}{dx}y(x)\right)\left(\frac{d}{dx}g(x)\right)\\
&\frac{d^2}{dx^2}g(x)+\frac{43139323}{7983360}\,g(x)\,y(x)\left(\frac{d}{dx}g(x)\right)^2\frac{d^2}{dx^2}g(x)\\
&+\frac{20495533}{3193344}\left(\frac{d}{dx}g(x)\right)y(x)\left(\frac{d^3}{dx^3}g(x)\right)\\
&\frac{d^2}{dx^2}g(x)+\frac{18173093}{5322240}\,(g(x))^2\,y(x)\left(\frac{d}{dx}g(x)\right)\\
&\left.\frac{d^3}{dx^3}g(x)\right]
\end{aligned}
\tag{72}
$$

The new proposed methods developed in paragraphs 3.2 and 3.3

$$
\begin{aligned}
\mathrm{LTE}_{\mathrm{PLD123b}} = h^{12}\Bigg[& G^3\Bigg[\frac{58061}{177408}\left(\frac{d}{dx}g(x)\right)^2 y(x)\\
&+\frac{58061}{133056}\,g(x)\,y(x)\,\frac{d^2}{dx^2}g(x)\\
&+\frac{1335403}{2661120}\left(\frac{d^4}{dx^4}g(x)\right)y(x)+\frac{58061}{266112}\left(\frac{d^3}{dx^3}g(x)\right)\frac{d}{dx}y(x)\Bigg]\\
&+G^2\Bigg[\frac{13876579}{31933440}\left(\frac{d^6}{dx^6}g(x)\right)y(x)+\frac{18173093}{5322240}\left(\frac{d}{dx}g(x)\right)y(x)\,\frac{d^3}{dx^3}g(x)\\
&+\frac{290305}{266112}\,g(x)\left(\frac{d}{dx}y(x)\right)\frac{d^3}{dx^3}g(x)+\frac{9115577}{15966720}\left(\frac{d^5}{dx^5}g(x)\right)\frac{d}{dx}y(x)\\
&+\frac{58061}{2128896}\,(g(x))^4\,y(x)+\frac{754793}{532224}\,g(x)\,y(x)\left(\frac{d}{dx}g(x)\right)^2\\
&+\frac{69731261}{31933440}\left(\frac{d^2}{dx^2}g(x)\right)^2 y(x)+\frac{1103159}{1064448}\,(g(x))^2\,y(x)\,\frac{d^2}{dx^2}g(x)\\
&+\frac{58061}{177408}\,(g(x))^2\left(\frac{d}{dx}y(x)\right)\frac{d}{dx}g(x)+\frac{2148257}{1330560}\,g(x)\,y(x)\,\frac{d^4}{dx^4}g(x)\\
&+\frac{290305}{133056}\left(\frac{d}{dx}g(x)\right)\left(\frac{d}{dx}y(x)\right)\frac{d^2}{dx^2}g(x)\Bigg]\\
&+G\Bigg[\frac{1335403}{15966720}\left(\frac{d^8}{dx^8}g(x)\right)y(x)
\end{aligned}
$$

$$+\frac{1335403}{399168}\left(\frac{d}{dx}g(x)\right)\left(\frac{d}{dx}y(x)\right)\frac{d^4}{dx^4}g(x)$$
$$+\frac{13876579}{15966720}g(x)\,y(x)\frac{d^6}{dx^6}g(x)+\frac{58061}{5322240}(g(x))^5\,y(x)$$
$$+\frac{18753703}{7983360}\left(\frac{d}{dx}g(x)\right)y(x)\frac{d^5}{dx^5}g(x)+\frac{58061}{249480}\left(\frac{d^7}{dx^7}g(x)\right)\frac{d}{dx}y(x)$$
$$+\frac{2148257}{1330560}(g(x))^2\,y(x)\frac{d^4}{dx^4}g(x)+\frac{69731261}{15966720}g(x)\,y(x)\left(\frac{d^2}{dx^2}g(x)\right)^2$$
$$+\frac{4238453}{798336}\left(\frac{d^2}{dx^2}g(x)\right)\left(\frac{d}{dx}y(x)\right)\frac{d^3}{dx^3}g(x)$$
$$+\frac{18173093}{2661120}g(x)\,y(x)\left(\frac{d^3}{dx^3}g(x)\right)\frac{d}{dx}g(x)$$
$$+\frac{2496623}{997920}\left(\frac{d^3}{dx^3}g(x)\right)^2y(x)+\frac{43139323}{7983360}\left(\frac{d}{dx}g(x)\right)^2y(x)\frac{d^2}{dx^2}g(x)$$
$$+\frac{754793}{181440}\left(\frac{d^2}{dx^2}g(x)\right)y(x)\frac{d^4}{dx^4}g(x)$$
$$+\frac{9115577}{7983360}g(x)\left(\frac{d}{dx}y(x)\right)\frac{d^5}{dx^5}g(x)$$
$$+\frac{290305}{66528}g(x)\left(\frac{d}{dx}y(x)\right)\left(\frac{d^2}{dx^2}g(x)\right)\frac{d}{dx}g(x)$$
$$+\frac{1103159}{1596672}(g(x))^3\,y(x)\frac{d^2}{dx^2}g(x)+\frac{290305}{266112}\left(\frac{d}{dx}g(x)\right)^3\frac{d}{dx}y(x)$$
$$+\frac{290305}{266112}(g(x))^2\left(\frac{d}{dx}y(x)\right)\frac{d^3}{dx^3}g(x)$$
$$+\frac{754793}{532224}(g(x))^2\,y(x)\left(\frac{d}{dx}g(x)\right)^2$$
$$+\frac{58061}{266112}(g(x))^3\left(\frac{d}{dx}y(x)\right)\frac{d}{dx}g(x)\bigg]$$
$$+\frac{290305}{266112}g(x)\left(\frac{d}{dx}y(x)\right)\left(\frac{d}{dx}g(x)\right)^3$$
$$+\frac{754793}{1596672}(g(x))^3\,y(x)\left(\frac{d}{dx}g(x)\right)^2$$
$$+\frac{58061}{1064448}(g(x))^4\left(\frac{d}{dx}y(x)\right)\frac{d}{dx}g(x)+\frac{58061}{31933440}(g(x))^6\,y(x)$$
$$+\frac{58061}{152064}\left(\frac{d^4}{dx^4}g(x)\right)^2y(x)+\frac{290305}{133056}(g(x))^2\left(\frac{d}{dx}y(x)\right)\left(\frac{d^2}{dx^2}g(x)\right)$$

$$\frac{d}{dx}g(x) + \frac{18173093}{5322240}(g(x))^2 y(x)\left(\frac{d^3}{dx^3}g(x)\right)$$

$$\frac{d}{dx}g(x) + \frac{18753703}{7983360}g(x)y(x)\left(\frac{d^5}{dx^5}g(x)\right)\frac{d}{dx}g(x)$$

$$+\frac{1335403}{399168}g(x)\left(\frac{d}{dx}y(x)\right)\left(\frac{d^4}{dx^4}g(x)\right)\frac{d}{dx}g(x)$$

$$+\frac{58061}{31933440}\left(\frac{d^{10}}{dx^{10}}g(x)\right)y(x) + \frac{58061}{3193344}\left(\frac{d^9}{dx^9}g(x)\right)\frac{d}{dx}y(x)$$

$$+\frac{69731261}{31933440}(g(x))^2 y(x)\left(\frac{d^2}{dx^2}g(x)\right)^2$$

$$+\frac{754793}{181440}g(x)y(x)\left(\frac{d^4}{dx^4}g(x)\right)\frac{d^2}{dx^2}g(x)$$

$$+\frac{4238453}{798336}g(x)\left(\frac{d}{dx}y(x)\right)\left(\frac{d^3}{dx^3}g(x)\right)\frac{d^2}{dx^2}g(x)$$

$$+\frac{43139323}{7983360}g(x)y(x)\left(\frac{d^2}{dx^2}g(x)\right)\left(\frac{d}{dx}g(x)\right)^2$$

$$+\frac{20495533}{3193344}\left(\frac{d}{dx}g(x)\right)y(x)\left(\frac{d^3}{dx^3}g(x)\right)$$

$$\frac{d^2}{dx^2}g(x) + \frac{290305}{236544}\left(\frac{d^2}{dx^2}g(x)\right)^3 y(x)$$

$$+\frac{9115577}{15966720}(g(x))^2\left(\frac{d}{dx}y(x)\right)\frac{d^5}{dx^5}g(x)$$

$$+\frac{1103159}{6386688}(g(x))^4 y(x)\frac{d^2}{dx^2}g(x)$$

$$+\frac{290305}{798336}(g(x))^3\left(\frac{d}{dx}y(x)\right)\frac{d^3}{dx^3}g(x) + \frac{58061}{114048}\left(\frac{d}{dx}g(x)\right)^4 y(x)$$

$$+\frac{6328649}{1596672}\left(\frac{d}{dx}g(x)\right)^2\left(\frac{d}{dx}y(x)\right)\frac{d^3}{dx^3}g(x)$$

$$+\frac{58061}{25344}\left(\frac{d}{dx}g(x)\right)^2 y(x)\frac{d^4}{dx^4}g(x)$$

$$+\frac{1103159}{591360}\left(\frac{d^2}{dx^2}g(x)\right)\left(\frac{d}{dx}y(x)\right)\frac{d^5}{dx^5}g(x)$$

$$+\frac{406427}{456192}\left(\frac{d}{dx}g(x)\right)\left(\frac{d}{dx}y(x)\right)\frac{d^6}{dx^6}g(x)$$

$$+\frac{58061}{22176}\left(\frac{d^3}{dx^3}g(x)\right)\left(\frac{d}{dx}y(x)\right)\frac{d^4}{dx^4}g(x)$$

$$
\begin{aligned}
&+\frac{13876579}{31933440}\,(g(x))^2\,y(x)\,\frac{d^6}{dx^6}g(x)\\
&+\frac{1799891}{354816}\left(\frac{d}{dx}g(x)\right)\left(\frac{d}{dx}y(x)\right)\left(\frac{d^2}{dx^2}g(x)\right)^2\\
&+\frac{2148257}{3991680}\,(g(x))^3\,y(x)\,\frac{d^4}{dx^4}g(x)\\
&+\frac{2496623}{997920}\,g(x)\,y(x)\left(\frac{d^3}{dx^3}g(x)\right)^2\\
&+\frac{754793}{3193344}\left(\frac{d}{dx}g(x)\right)y(x)\,\frac{d^7}{dx^7}g(x)\\
&+\frac{1799891}{2661120}\left(\frac{d^3}{dx^3}g(x)\right)y(x)\,\frac{d^5}{dx^5}g(x)\\
&+\frac{987037}{2128896}\left(\frac{d^2}{dx^2}g(x)\right)y(x)\,\frac{d^6}{dx^6}g(x)\\
&+\frac{58061}{249480}\,g(x)\left(\frac{d}{dx}y(x)\right)\frac{d^7}{dx^7}g(x)\\
&+\frac{1335403}{15966720}\,g(x)\,y(x)\,\frac{d^8}{dx^8}g(x)\Bigg]
\end{aligned}
\tag{73}
$$

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